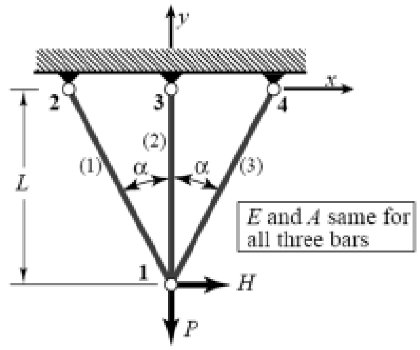
Assignment 1:



1)

First we compute the local stiffness matrices in global coordinate:

- Bar 1:

Bar 1 is rotated an angle $\theta_1 = \alpha + 90º$:

$$\cos(\theta_1) = -\sin(\alpha) = -s$$
$$\sin(\theta_1) = \cos(\alpha) = c$$

The length of the bar is:

$$L_1 = \frac{L}{\cos(\alpha)} = \frac{L}{c}$$

The local stiffness matrix expressed in the global coordinates is:

$$K_1 = \frac{EA}{L}\begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ -c^2s & c^3 & c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s \\ c^2s & -c^3 & -c^2s & c^3 \end{bmatrix}$$

- Bar 2:

Bar 2 is rotated an angle $\theta_2 = 90º$:

$$\cos(\theta_2) = 0$$
$$\sin(\theta_2) = 1$$

The length of the bar is:

$$L_2 = L$$

The local stiffness matrix expressed in the global coordinates is:

$$K_2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

- Bar 3:

Bar 3 is rotated an angle $\theta_3 = 90º - \alpha$:

$$\cos(\theta_3) = \sin(\alpha) = s$$
$$\sin(\theta_3) = \cos(\alpha) = c$$

The length of the bar is:

$$L_3 = \frac{L}{\cos(\alpha)} = \frac{L}{c}$$

The local stiffness matrix expressed in the global coordinates is:

$$K_3 = \frac{EA}{L}\begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -c^3 & c^2s & c^3 \end{bmatrix}$$

Assembling the global matrix:

$$K_1 = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 2c^3+1 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}$$

The only external forces acting over the structure are H and P. The system of equations to be solved is:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 2c^3+1 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The 5th row and column contain only zeros because they are related to the horizontal displacement and force on node 3. Since the forces in the bars are assumed to be only axial and bar 2 is a vertical bar, there can be no force in the horizontal direction in that bar.

2)

The boundary conditions are the following:

- Dirichlet bc:

$$\begin{cases} u_{x2} = 0 \\ u_{y2} = 0 \\ u_{x3} = 0 \\ u_{y3} = 0 \\ u_{x4} = 0 \\ u_{y4} = 0 \end{cases}$$

- Forces: Vertical force –P and horizontal force H in node 1.

After applying the boundary conditions, the reduced system of equations is obtained:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 2c^3+1 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

3)

Solving the reduced system of equations:

$$\begin{cases} u_{x1} = \dfrac{HL}{2cs^2EA} \\ u_{y1} = -\dfrac{PL}{(2c^3+1)EA} \end{cases}$$

For $\alpha \to 0, \sin(\alpha) = s = 0$ and $\cos(\alpha) = c = 1$. In this case, the structure is made only of vertical bars and the bars cannot hold horizontal forces. Then, if $H \neq 0$, $u_{x1} \to \infty$. The problem in the vertical direction can be considered as a bar with area 3A and an axial force P. Thus, the vertical displacement of node 1 will be $u_{y1} = -\frac{PL}{3EA}$, which is the result obtained.

For $\alpha \to \frac{\pi}{2}, \cos(\alpha) = c = 0$ and $\sin(\alpha) = s = 1$. In this case, bars 2 and 3 become horizontal bars with infinite length and the only bar that can have vertical forces is bar 2. The problem in the vertical directiong can be considered as a bar with area A and length L and an axial force P. Thus, the vertical displacement obtained will be $u_{y1} = -\frac{PL}{EA}$, which is the result obtained.

4)

The axial forces are computed from the horizontal displacements of the bars in the local coordinate system:

- Bar 1:

$$\bar{u}_{x1}^{1} = -u_{x1}\sin(\alpha) + u_{y1}\cos(\alpha) = -\frac{HL}{2csEA} - \frac{PLc}{(2c^3+1)EA}$$

$$\bar{u}_{x2}^{1} = 0$$

The elongation of the bar is:

$$d^1 = \bar{u}_{x2}^{1} - \bar{u}_{x1}^{1} = \frac{HL}{2csEA} + \frac{PLc}{(2c^3+1)EA}$$

The force in bar 1 is:

$$F^1 = \frac{EA}{L^1}d^1 = \frac{EAc}{L}d^1 = \frac{H}{2s} + \frac{Pc^2}{2c^3+1}$$

- Bar 2:

$$\bar{u}_{x1}^{2} = u_{y1} = -\frac{PL}{(2c^3+1)EA}$$

$$\bar{u}_{x2}^{2} = 0$$

The elongation of the bar is:

$$d^2 = \bar{u}_{x2}^{2} - \bar{u}_{x1}^{2} = \frac{PL}{(2c^3+1)EA}$$

The force in bar 2 is:

$$F^2 = \frac{EA}{L^2}d^2 = \frac{EA}{L}d^2 = \frac{P}{2c^3+1}$$

- Bar 3:

$$\bar{u}_{x1}^{3} = u_{x1}\sin(\alpha) + u_{y1}\cos(\alpha) = \frac{HL}{2csEA} - \frac{PLc}{(2c^3+1)EA}$$

$$\bar{u}_{x2}^{3} = 0$$

The elongation of the bar is:

$$d^3 = \bar{u}_{x2}^{3} - \bar{u}_{x1}^{3} = -\frac{HL}{2csEA} + \frac{PLc}{(2c^3+1)EA}$$

The force in bar 3 is:

$$F^3 = \frac{EA}{L^3}d^3 = \frac{EAc}{L}d^3 = -\frac{H}{2s} + \frac{Pc^2}{2c^3+1}$$

For the limit case where $\alpha \to 0$ ($\sin(\alpha) = s = 0$ and $\cos(\alpha) = c = 1$), the structure is only made of vertical bars and cannot hold the horizontal force H, which should be hold by bars 1 and 3. Then, if $\alpha \to 0$ and $H \neq 0$, the forces F^1 and F^3 "blow up" (they tend to infinite as they have a term divided by 0).