CSMD: Assignment 10: Dynamics

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Question 1

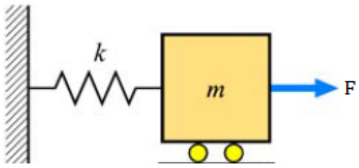

Figure 1

The system depicted in figure 1 can be modeled as:

$$ku - m\frac{d^2u}{dt^2} = F$$

The effects of F can be seen in figure 2, which has been obtained for $k = 2$,$m = 1$. f_n is defined as $\sqrt{\frac{k}{m}}$. As can be seen, when a constant force is applied, the system oscillates with the natural frequency $\frac{\sqrt{f_n}}{2\pi} = 0.225$ (a whole cycle every 4.44 s). However, for a sinusoidal load with the natural frequency of the structure, the result is unbounded and tends to increase its amplitude without control.

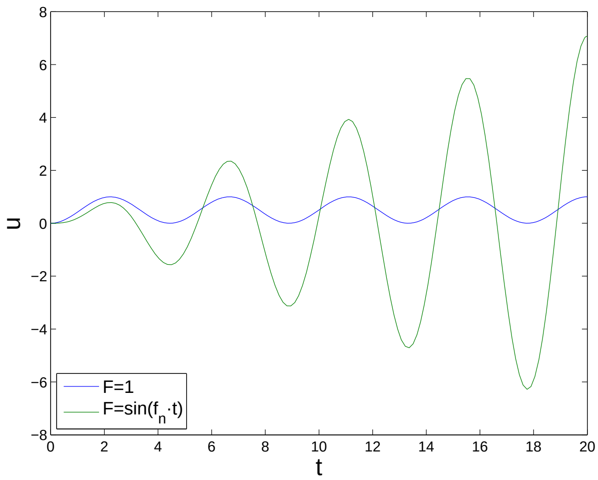

Figure 2

Question 2

A uniform axial bar with length L and negligible mass which is clamped at both ends and has a weight of mass m placed at its center is subjected to three forces : The upper reaction R_2 (at $y = L$), the reaction at the bottom R_1 (at $y = 0$) and the force due to the mass placed at $y = 0$ which is mg. R_1 and R_2 can be obtained from equilibrium of forces and compatibility condition:

$$\sum F = 0 \quad \rightarrow \quad mg = R_1 - R_2$$

$$u(0) = u(L) = 0 \quad \rightarrow \quad 0 = \frac{R_1 L}{EA} + \frac{R_1 L}{EA} \quad \rightarrow \quad R_1 = -R_2 = \frac{mg}{2}$$

The displacement at $y = \frac{L}{2}$ is :

$$u_{\frac{L}{2}} = \frac{R_1 L}{EA} = \frac{mgL}{2EA}$$

The stiffness K is then:

$$K = \frac{mg}{u_{\frac{L}{2}}} = \frac{2EA}{L}$$

Since we are neglecting the mass of the bar, the total mass of the system is m. Thus, the natural frequency of vibration can be expressed as:

$$f_n = \sqrt{2\frac{EA}{mL}}$$

Question 3

The consistent element mass matrix is calculated as:

$$m = \int_{\Omega^e} \boldsymbol{N^T N} \rho dV$$

Using an isoparametric representation of the two-node bar element:

$$m = \rho A \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} |J| d\xi =$$

$$= \rho \frac{AL}{8} \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} d\xi = \rho \frac{AL}{8} \int_{-1}^{1} \begin{bmatrix} 1 - 2\xi + \xi^2 & 1 - \xi^2 \\ 1 - \xi^2 & 1 + 2\xi + \xi^2 \end{bmatrix} d\xi =$$

$$= \rho \frac{AL}{8} \begin{bmatrix} \xi - \xi^2 + \frac{\xi^3}{3} & \xi - \frac{\xi^3}{3} \\ \xi - \frac{\xi^3}{3} & \xi + \xi^2 + \frac{\xi^3}{3} \end{bmatrix}_{-1}^{1} = \rho \frac{AL}{8} \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix} = \rho \frac{AL}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \boldsymbol{\rho \frac{AL}{3}} & \boldsymbol{\rho \frac{AL}{6}} \\ \boldsymbol{\rho \frac{AL}{6}} & \boldsymbol{\rho \frac{AL}{3}} \end{bmatrix}}$$

Question 4

We can express the variation of area as:

$$A(\xi) = \sum_{i=1}^{2} N_i(\xi) A_i = \frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi)$$

Thus:

$$m = \int_{-1}^{1} \boldsymbol{N^T N} \rho A |J| d\xi = \rho \frac{L}{2} \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \left(\frac{A_1}{2}(1-\xi) + \frac{A_2}{2}(1+\xi) \right) d\xi =$$

$$= \rho \frac{L}{16} \int_{-1}^{1} \begin{bmatrix} (1-\xi)^2 & (1-\xi)(1+\xi) \\ (1-\xi)(1+\xi) & (1+\xi)^2 \end{bmatrix} \left(A_1(1-\xi) + A_2(1+\xi) \right) d\xi =$$

$$= \rho \frac{L}{16} \int_{-1}^{1} A_1 \begin{bmatrix} (1-\xi)^3 & (1-\xi)^2(1+\xi) \\ (1-\xi)^2(1+\xi) & (1+\xi)^2(1-\xi) \end{bmatrix} + A_2 \begin{bmatrix} (1-\xi)^2(1+\xi) & (1-\xi)(1+\xi)^2 \\ (1-\xi)(1+\xi)^2 & (1+\xi)^3 \end{bmatrix} =$$

$$= \rho \frac{L}{16} \int_{-1}^{1} A_1 \begin{bmatrix} 1 - 3\xi + 3\xi^2 - \xi^3 & 1 - \xi - \xi^2 + \xi^3 \\ 1 - \xi - \xi^2 + \xi^3 & 1 + \xi - \xi^2 - \xi^3 \end{bmatrix} + A_2 \begin{bmatrix} 1 - \xi - \xi^2 + \xi^3 & 1 + \xi - \xi^2 - \xi^3 \\ 1 + \xi - \xi^2 - \xi^3 & 1 + 3\xi + 3\xi^2 + 3\xi^3 \end{bmatrix} =$$

$$= \rho \frac{L}{16} \left[A_1 \begin{bmatrix} \xi - \frac{3\xi^2}{2} + \xi^3 - \frac{\xi^4}{4} & \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} \\ \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \end{bmatrix} + A_2 \begin{bmatrix} \xi - \frac{\xi^2}{2} - \frac{\xi^3}{3} + \frac{\xi^4}{4} & \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} \\ \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} - \frac{\xi^4}{4} & \xi + \frac{3\xi^2}{2} + \xi^3 + \frac{\xi^4}{4} \end{bmatrix} \right]_{-1}^{1} =$$

$$= \boxed{\begin{bmatrix} \boldsymbol{\rho L \frac{3A1+A2}{12}} & \boldsymbol{\rho L \frac{A1+A2}{12}} \\ \boldsymbol{\rho L \frac{A1+A2}{12}} & \boldsymbol{\rho L \frac{A1+3A2}{12}} \end{bmatrix}}$$

Question 5

The simplest procedure to obtain the lumped mass matrix of the 3D 2-node bar is to assign half of the mass of the bar to every node:

$$m = \frac{\rho LA}{2} \boldsymbol{I_6} = \boxed{\begin{bmatrix} \boldsymbol{\frac{\rho LA}{2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\frac{\rho LA}{2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\frac{\rho LA}{2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\frac{\rho LA}{2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\frac{\rho LA}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\frac{\rho LA}{2}} \end{bmatrix}}$$