Assignment 4
Computational Structural Mechanics and Dynamics

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Assignment 4.1

The three-node bar element referred to the coordinate ξ follows the variations sketched in Fig. 1. These shape functions must be quadratic polynomials in ξ :

$$N_1^e(\xi) = a_0 + a_1\xi + a_1\xi^2 \; N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \; N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

The expression of N_1^e can be found using Lagrange interpolating polynomials. Knowing $N_1^e(-1) = 1$, $N_1^e(0) = 0$ and $N_1^e(1) = 0$:

$$N_1^e(\xi) = \frac{(\xi)(\xi-1)}{(-1)(-1-1)} \cdot 1 + \frac{(\xi+1)(\xi-1)}{(1)(-1)} \cdot 0 + \frac{(\xi+1)(\xi)}{(1+1)(1)} \cdot 0 = \frac{1}{2}(\xi)(\xi-1)$$

Doing the same for N_2^e and $N3^e$:

$$N_2^e(\xi) = \frac{(\xi)(\xi-1)}{(-1)(-1-1)} \cdot 0 + \frac{(\xi+1)(\xi-1)}{(1)(-1)} \cdot 1 + \frac{(\xi+1)(\xi)}{(1+1)(1)} \cdot 0 = \frac{1}{2}(\xi)(\xi+1)$$

$$N_3^e(\xi) = \frac{(\xi)(\xi-1)}{(-1)(-1-1)} \cdot 0 + \frac{(\xi+1)(\xi-1)}{(1)(-1)} \cdot 0 + \frac{(\xi+1)(\xi)}{(1+1)(1)} \cdot 1 = 1 - \xi^2$$

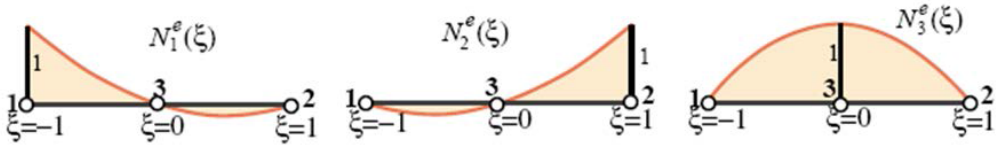

Figure 1: Isoparametric shape functions for 3-node bar element.

The sum of the 3 shape functions must be equal to 1:

$$N_1^e(\xi)+N_2^e(\xi)+N_2^e(\xi) = \frac{1}{2}(\xi)(\xi+1)+\frac{1}{2}(\xi)(\xi-1)+1-\xi^2 = \xi^2-\xi^2+\frac{\xi-\xi}{2}+1 = 1$$

Assignment 4.2

1)

The isoparametric definition of the 3-node straight bar element (Fig. 2) is:

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Where $x_1 = 0$,$x_2 = l$ and $x_3 = \left(\frac{1}{2}+\alpha\right) l$, being $-\frac{1}{2} < \alpha < \frac{1}{2}$. The Jacobian can be obtained as:

$$J = \frac{dx}{d\xi} = \sum_{i=1}^{3} x_i \frac{dN_i^e}{d\xi} = 0 \cdot \left(\xi - \frac{1}{2}\right) + l\left(\xi+\frac{1}{2}\right)+\left(\frac{1}{2}+\alpha\right) l\,(-2\xi) = l\left(\frac{1-4\xi\alpha}{2}\right)$$

Thus:

a)

$$\text{For } \begin{cases} -\frac{1}{4} < \alpha < \frac{1}{4} \\ -1 \le \xi \le 1 \end{cases} \quad \to 1-4\xi\alpha > 0 \to J > 0$$

b)

$$\text{For } \alpha = 0 \to J = l\left(\frac{1-4\xi\cdot 0}{2}\right) = \frac{l}{2}$$

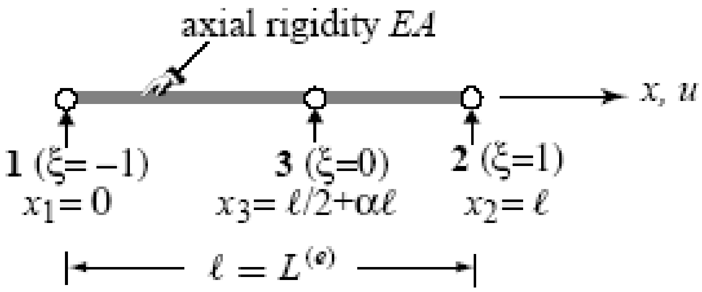

Figure 2: 3-node bar element

2)

The elemental strains and nodal displacements are related by the strain matrix **B**:

$$e = \boldsymbol{B}\boldsymbol{u}^{\boldsymbol{e}}$$

The strain displacement matrix is expressed as:

$$\boldsymbol{B} = \frac{d\boldsymbol{N}}{dx} = J^{-1}\frac{d\boldsymbol{N}}{d\xi} = \frac{2}{l\left(1-4\xi\alpha\right)}\begin{bmatrix}\frac{dN_1^e}{d\xi} & \frac{dN_2^e}{d\xi} & \frac{dN_3^e}{d\xi}\end{bmatrix} = \frac{2}{l\left(1-4\xi\alpha\right)}\begin{bmatrix}\xi-\frac{1}{2} & \xi+\frac{1}{2} & -2\xi\end{bmatrix}$$

3)

The elemental stiffness matrix obtained from the Minimum Potential Energy Principle for a bar element is:

$$\boldsymbol{K}^e = \int_0^l EA\boldsymbol{B}^T\boldsymbol{B}dx$$

We can change the integration variable using $dx = Jd\xi$ and changing the limits of integration taking into account that to go from x to ξ, $x_1 = 0 \rightarrow \xi_1 = -1$ and $x_2 = l \rightarrow \xi_2 = 1$:

$$\boldsymbol{K}^e = \int_0^l EA\boldsymbol{B}^T\boldsymbol{B}dx = \int_{-1}^1 EA\boldsymbol{B}^T\boldsymbol{B}Jd\xi$$

Assignment 4.3

A five node quadrilateral element has the nodal configuration shown in Fig. 3. The shape functions of this element must satisfy compatibility and verify that their sum is unity. In order to find them, we must find first $N_5\left(\xi,\eta\right)$. It is known that N_5 must be 0 at the rest of the nodes and edges of the element ($\xi=-1, \xi=1, \eta=-1$ and $\eta=1$) and 1 at $(\xi,\eta)=(0,0)$:

$$N_5\left(\xi,\eta\right) = 1\frac{\left(\xi+1\right)\left(\xi-1\right)\left(\eta+1\right)\left(\eta-1\right)}{\left(0+1\right)\left(0-1\right)\left(0+1\right)\left(0-1\right)} = \left(\xi^2-1\right)\left(\eta^2-1\right)$$

The corner shape functions can be obtained from the shape functions of the 4-node quadrilateral ($\underline{N_1}, \underline{N_2}, \underline{N_3}$ and $\underline{N_4}$) but adding a correction αN_5 so that all N_i vanish at node 5:

$$N_1 = \underline{N_1} + \alpha N_5 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right) + \alpha N_5$$

$$N_2 = \underline{N_2} + \alpha N_5 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right) + \alpha N_5$$

$$N_3 = \underline{N_3} + \alpha N_5 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right) + \alpha N_5$$

$$N_4 = \underline{N_4} + \alpha N_5 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right) + \alpha N_5$$

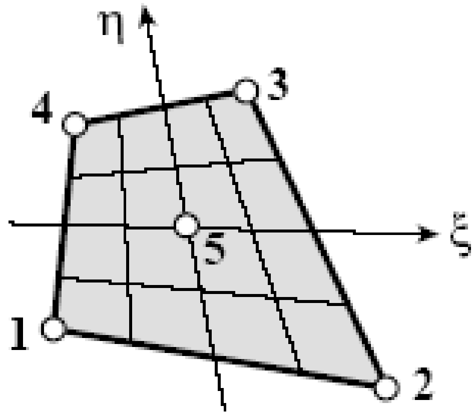

Figure 3: 5-node quadrilateral element

For $(\xi,\eta)=(0,0) \rightarrow \begin{cases} \underline{N_1} = \underline{N_2} = \underline{N_3} = \underline{N_4} = \frac{1}{4} \\ N_5 = 1 \end{cases}$. Thus, for $N_1, N_2 N_3$ and N_4 to vanish at $N_5 \rightarrow \boldsymbol{\alpha = -\frac{1}{4}}$.
The shape functions of the 5 nodes quadrilateral element are then:

$$
\begin{aligned}
N_1(\xi,\eta) &= \frac{1}{4}\left[(1-\xi)(1-\eta) - \left(\xi^2-1\right)\left(\eta^2-1\right)\right] \\
N_2(\xi,\eta) &= \frac{1}{4}\left[(1+\xi)(1-\eta) - \left(\xi^2-1\right)\left(\eta^2-1\right)\right] \\
N_3(\xi,\eta) &= \frac{1}{4}\left[(1+\xi)(1+\eta) - \left(\xi^2-1\right)\left(\eta^2-1\right)\right] \\
N_4(\xi,\eta) &= \frac{1}{4}\left[(1-\xi)(1+\eta) - \left(\xi^2-1\right)\left(\eta^2-1\right)\right] \\
N_5(\xi,\eta) &= \left(\xi^2-1\right)\left(\eta^2-1\right)
\end{aligned}
$$

The sum of the shape functions must be 1:

$$
\begin{aligned}
N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}&[(1-\xi)(1-\eta)+(1+\xi)(1-\eta)+(1+\xi)(1+\eta)+ \\
&+(1-\xi)(1+\eta)] + \left(1-\frac{4}{4}\right)\left[\left(\xi^2-1\right)\left(\eta^2-1\right)\right] = 1
\end{aligned}
$$