Polytechnic University of Catalonia

Msc. in Numerical Methods

Computational Structural Mechanics and Dynamics
Assignment 3.1:"The Plane Stress Problem"
Assignment 3.2: "The 3-node Plane Stress Triangle"

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February 29, 2016

Assignment 3.1: "The Plane Stress Problem"

(a)

For the plane stress problem, we call $E = E*$ and $\nu = \nu*$. Making the stresses σ_{xx} and σ_{yy} equal for both plane stress and plane strain, we obtain the following system of equations

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[e_{xx} + \frac{\nu}{1-\nu}e_{yy}\right] = \frac{E*}{1-\nu*^2}\left[\nu * e_{xx} + e_{yy}\right]$$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[e_{yy} + \frac{\nu}{1-\nu}e_{xx}\right] = \frac{E*}{1-\nu*^2}\left[\nu * e_{yy} + e_{xx}\right]$$

Solving for $E*$ and $\nu*$ we obtain

$$E* = \frac{E}{1-\nu^2}$$
$$\nu* = \frac{\nu}{1-\nu}$$

Now, it is possible to go from one state to the other using this two fictitious modulus.

From plane stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E*}{1-\nu*^2}\begin{bmatrix} 1 & \nu* & 0 \\ \nu* & 1 & 0 \\ 0 & 0 & \frac{1-\nu*}{2} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

Replacing $E* = \frac{E}{1-\nu^2}$ and $\nu* = \frac{\nu}{1-\nu}$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{\frac{E}{1-\nu^2}}{(1-(\frac{\nu}{1-\nu})^2)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{\nu}{1-\nu}}{2} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

Rearranging the fractions we end up with the plane strain system

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

Then, to go back to plane stress we just have to invert the previous results $E* = \frac{E(2\nu+1)}{(\nu+1)^2}$ and $\nu* = \frac{\nu}{\nu+1}$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{\frac{E(2\nu+1)}{(\nu+1)^2}(1-\frac{\nu}{\nu+1})}{(1+\frac{\nu}{\nu+1})(1-2\frac{\nu}{\nu+1})} \begin{bmatrix} 1 & \frac{\frac{\nu}{\nu+1}}{1-\frac{\nu}{\nu+1}} & 0 \\ \frac{\frac{\nu}{\nu+1}}{1-\frac{\nu}{\nu+1}} & 1 & 0 \\ 0 & 0 & \frac{1-2\frac{\nu}{\nu+1}}{2(1-\frac{\nu}{\nu+1})} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

When simplified, the expression is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

(b)

Following Hook's law $\sigma = eE$ and its inverse $e = E^{-1}\sigma$
Then

$$U = \frac{1}{2}e^t E e = \frac{1}{2}\sigma^t e = \frac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy}\sigma_{xy}e_{xy})$$
$$U = \frac{1}{2}\sigma^t E^{-1}\sigma = \frac{1}{2}\sigma^t e = \frac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy}\sigma_{xy}e_{xy})$$

Assignment 3.2: "The 3-node Plane Stress Triangle"

(1)

Given the coordinates of the three points of the triangle

$x_1 = 0$, $x_2 = 3$, $x_3 = 2$, $y_1 = 0$, $y_2 = 1$, $y_3 = 2$

We can calculate the area of the triangle by the following determinant

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2$$

And the element strain matrix

$$B = \frac{1}{2A}\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0 \\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1 \\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

For a constant thickness $h = 1$, the element stiffness matrix can be calculated as $K_e = hB^T EB$

$$K_e = \frac{1}{8}\begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \\ 3 & 0 & -1 \\ -1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$Ke = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}$$

(2)

The sum of rows (or columns) are

$$\begin{aligned} R_1 &= 18.75 + 9.375 - 12.5 - 6.25 - 6.25 + -3.125 = 0 \\ R_2 &= 9.375 + 18.75 + 6.25 + 12.5 - 15.625 - 31.25 = 0 \\ R_3 &= -12.5 + 6.25 + 75 - 37.5 - 62.5 + 31.25 = 0 \\ R_4 &= -6.25 + 12.5 - 37.5 + 75 + 43.75 - 87.5 = 0 \\ R_5 &= -6.25 - 15.625 - 62.5 + 43.75 + 68.75 - 28.125 = 0 \\ R_6 &= -3.125 - 31.25 + 31.25 - 87.5 - 28.125 + 118.75 = 0 \end{aligned}$$

The fact that the sum of $R1 + R3 + R5$ and $R2 + R4 + R6$ is equal to zero can be explained by the equilibrium of the forces in the triangle.

For the x coordinate

$$\begin{aligned} K_{11}u_{x1} + K_{12}u_{x2} + K_{13}u_{x3} + K_{14}u_{x4} + K_{15}u_{x5} + K_{16}u_{x6} &= f_{x1} \\ K_{31}u_{x1} + K_{32}u_{x2} + K_{33}u_{x3} + K_{34}u_{x4} + K_{35}u_{x5} + K_{36}u_{x6} &= f_{x2} \\ K_{51}u_{x1} + K_{52}u_{x2} + K_{53}u_{x3} + K_{54}u_{x4} + K_{55}u_{x5} + K_{56}u_{x6} &= f_{x3} \end{aligned}$$

Summing the tree equations, we can apply equilibrium of forces in x direction to be zero

$$\begin{aligned} K_{11}u_{x1} + K_{12}u_{x2} + K_{13}u_{x3} + K_{14}u_{x4} + K_{15}u_{x5} + K_{16}u_{x6} + \\ K_{31}u_{x1} + K_{32}u_{x2} + K_{33}u_{x3} + K_{34}u_{x4} + K_{35}u_{x5} + K_{36}u_{x6} + \\ K_{51}u_{x1} + K_{52}u_{x2} + K_{53}u_{x3} + K_{54}u_{x4} + K_{55}u_{x5} + K_{56}u_{x6} \\ = f_{x1} + f_{x2} + f_{x3} = 0 \end{aligned}$$

It is evident that for any rigid displacement (because the non-existence of boundary conditions the resulting force should be zero, then the sum of the rigidity terms should also be zero.

The same explanations is valid for the y direction and columns 2,4,6.