# CSMD Assignment 1

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## 1 Show master stiffness equations

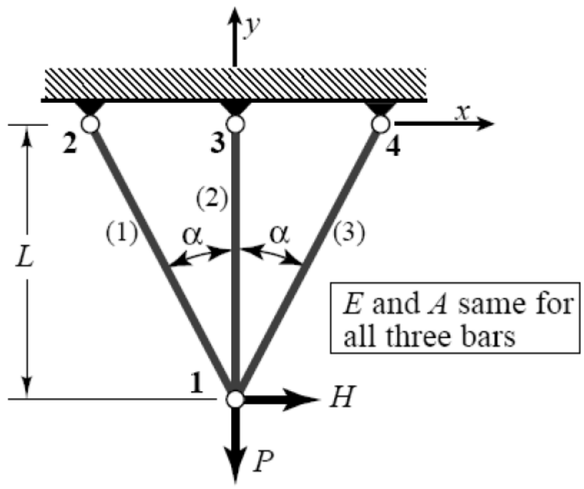


Figure 1: Statically indeterminate structure

The Hook law for a bar in the local reference system reads

$$\bar{\mathbf{f}}^e = \bar{\mathbf{K}}^e \bar{\mathbf{u}}^e$$

and, explicitly

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

where $l$ equals $L$ for bar (2) and it equals $L/\cos(\alpha)$ for bars (1) and (3).

With the aim to put together all the bars in the truss system, a the transformation is defined depending on the $\alpha$ angle measured counter-clockwise from the vertical. The nomenclature defined in the Assignment wording for $c$ and $s$ is followed.

In matrix form

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e$$

$$\bar{\mathbf{f}}^e = \mathbf{T}^e \mathbf{f}^e$$

Bar (1) contributes to the stiffness equations with

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x2} \\ \bar{u}_{y2} \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

Bar (2) contributes to the stiffness equations with

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x3} \\ \bar{u}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Bar (3) contributes to the stiffness equations with

$$\begin{bmatrix} \bar{u}_{x1} \\ \bar{u}_{y1} \\ \bar{u}_{x4} \\ \bar{u}_{y4} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Therefore, the Hook law in global reference system using the local expression of the stiffness matrix reads

$$\mathbf{f}^e = (\mathbf{T}^e)^T \bar{\mathbf{K}}^e \mathbf{T}^e \mathbf{u}^e$$

The elemental Hook law in global system result

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L}c \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ -cs & -c^2 & -cs & c^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L}c \begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Finally, transforming the elemental stiffness matrices in contributions to the global stiffness matrix by adding the missing DOFs and adding up the three contributions, yields

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ 0 & 2c^3+1 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ c^2s & -c^3 & -c^2s & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -cs^2 & -c^2s & 0 & 0 & 0 & 0 & cs^2 & c^2s \\ -c^2s & -c^3 & 0 & 0 & 0 & 0 & c^2s & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

The $5^{th}$ DOF is null and doesn't contribute to the solution. From the point of view of the "null row", it can be interpreted as: the displacement in x direction of node 3 is always zero no matter the elastic state of the rest of DOFs. That is because, in a truss, the bars can only transmit forces in axial direction and node 3 doesn't link any bar with x component direction. From the point of view of the "null column", this result can be interpreted as: The equilibrium of any DOF depends on the state of the rest of DOFs (their displacement) except from the displacement of the $5^{th}$ DOF because its value is irrelevant due to geometrical configuration.

## 2 Apply BCs and show the 2-equation modified stiffness system

The two first DOFs of the truss can be transformed as follows without changing its physical meaning:

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 2c^3+1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} - \frac{EA}{L} \begin{bmatrix} -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Boundary conditions set displacements of nodes 2, 3 and 4 to zero and can be expressed as

$$\mathbf{u}_i = \mathbf{0} \qquad i = 2, 3, 4$$

Taking on account the above boundary conditions the modified stiffness system reads

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 2c^3+1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

## 3 Solve for the displacements $u_{x1}$ and $u_{y1}$

Solving for the displacements

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} \frac{1}{2cs^2} & 0 \\ 0 & \frac{1}{2c^3+1} \end{bmatrix} \begin{bmatrix} H \\ -P \end{bmatrix}$$

$$u_{x1} = \frac{L}{EA} \frac{1}{2cs^2} H$$

$$u_{y1} = -\frac{L}{EA}\frac{1}{2c^3+1}P$$

In the limit case when $\alpha \to 0$, other quantities change as follows: $c \to 1$, $s \to 0$

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} \to \frac{L}{EA}\begin{bmatrix} \infty & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} H \\ -P \end{bmatrix}$$

Therefore,

$$u_{x1} \to \infty$$

$$u_{y1} \to -\frac{1}{3}\frac{PL}{EA}$$

The $x$ component compliance diverges and therefore a finite value of $H$ produces a infinite displacement $u_{x1}$. At the view of these results, statics is not observed anymore and the alignment of the three elements confers the system with a kinematic degree of freedom.

The $y$ component compliance is one third that of the bar of length $L$. This makes sense as (in absence of load $H$) load $P$ is shared among three equal bars of length $L$.

In the limit case when $\alpha \to \frac{\pi}{2}$, other quantities change as follows: $c \to 0$, $s \to 1$

$$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} \to \frac{L}{EA}\begin{bmatrix} \infty & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H \\ -P \end{bmatrix}$$

With $\alpha \to \frac{\pi}{2}$ the length of bars (1) and (2) diverge to infinity and their axial stiffness collapses to zero. Therefore horizontal compliance vanishes and a similar analysis can be made for displacement $u_{x1}$ as that made for $\alpha \to 0$

For similar reasons, bars (1) and (2) do not contribute with stiffness in vertical direction and $y$ component stiffness is that of the bar of length $L$. This makes sense as (in absence of load $H$) load $P$ is bore by bar (2) alone.

# 4 Recover the axial forces in the three members

Using the Superposition Principle, the axial forces of each bar can be derived by projecting $u_{x1}$ and $u_{y1}$ on the bar direction. Therefore,

$$F_i = K_i \delta_i$$

$$F_1 = \frac{EA}{L}c\Big(\frac{L}{EA}\frac{s}{2cs^2}H - \frac{L}{EA}\frac{c}{2c^3+1}P\Big)$$

$$F_1 = \frac{1}{2s}H - \frac{c^2}{2c^3+1}P$$

$$F_2 = \frac{EA}{L}\Big(-\frac{L}{EA}\frac{-1}{2c^3+1}P\Big)$$

$$F_2 = \frac{1}{2c^3+1}P$$

$$F_3 = \frac{EA}{L}c\Big(\frac{L}{EA}\frac{-s}{2cs^2}H - \frac{L}{EA}\frac{-c}{2c^3+1}P\Big)$$

$$F_3 = -\frac{1}{2s}H + \frac{c^2}{2c^3+1}P$$

In the limit case when $\alpha \to 0$, $F_1$ and $F_3$ diverge because the system remains statically indeterminate (only when $\alpha = 0$ the system degenerates to the kinematic degree of freedom) and the horizontal force $H$ is cancelled out with the sum of the horizontal projections of $F_1$ and $F_3$. The smaller $\alpha$ is, the bigger $F_1$ and $F_3$ must be.