COMPUTATIONAL STRUCTURAL MECHANICS & DYNAMICS

Report on Assignment 1

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1. The Direct Stiffness Method The stiffness equation for a two-noded truss element is:

$$\mathbf{f}^{(e)} = \mathbf{K}^{(e)}\mathbf{u}^{(e)}$$

$$\text{where} \quad \mathbf{K}^{(e)} = \frac{EA}{L^{(e)}} \left[\begin{array}{cc|cc} c^2\phi & sc\phi & -c^2\phi & -sc\phi \\ sc\phi & s^2\phi & -sc\phi & -s^2\phi \\ \hline -c^2\phi & -cs\phi & c^2\phi & cs\phi \\ -sc\phi & -s^2\phi & sc\phi & s^2\phi \end{array}\right] = \left[\begin{array}{c|c} \mathbf{K}_{11}^{(e)} & \mathbf{K}_{12}^{(e)} \\ \hline \mathbf{K}_{21}^{(e)} & \mathbf{K}_{22}^{(e)} \end{array}\right]$$

where $L^{(e)}$ is length of the corresponding truss element, $c^m s^n \phi = cos^m\ \phi^{(e)} sin^n\ \phi^{(e)}$, with $\phi^{(e)}$ as the angle between element and x-axis.

For each element, $L^{(e)}$ and $\phi^{(e)}$ are:

(e)	$L^{(e)}$	$\phi^{(e)}$
1	$\frac{L}{cos\ \alpha}$	$\frac{\pi}{2} + \alpha$
2	L	$\frac{\pi}{2}$
3	$\frac{L}{cos\ \alpha}$	$\frac{\pi}{2} - \alpha$

(a) Using the assembly rules, the assembly matrix will be :

$$\mathbf{K}_a = \begin{bmatrix} \mathbf{K}_{11}^{(1)} + \mathbf{K}_{11}^{(2)} + \mathbf{K}_{11}^{(3)} & \mathbf{K}_{12}^{(1)} & \mathbf{K}_{12}^{(2)} & \mathbf{K}_{12}^{(3)} \\ \mathbf{K}_{21}^{(1)} & \mathbf{K}_{22}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{21}^{(2)} & \mathbf{0} & \mathbf{K}_{22}^{(2)} & \mathbf{0} \\ \mathbf{K}_{21}^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{21}^{(3)} \end{bmatrix}$$

$$\implies \mathbf{K}_a = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}$$

where $c = cos\alpha$ and $s = sin\alpha$.

The fifth row and column are zeros. This is because the fifth node is free to move in x-direction. It doesn't need any force to move in x-direction, NOR any force in x-direction influences the displacement of other nodes.

(b) The nodes 2,3 and 4 are fixed. Hence, we apply boundary conditions $u_{x2} = u_{y2} = u_{x3} = u_{y3} = u_{x4} = u_{y4} = 0$. Also, we use $f_{x1} = H$ and $f_{y1} = -P$ Hence, the reduced system of equations will be:

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix}$$

(c) The above system of equation yields $u_{x1} = \dfrac{L}{EA}\dfrac{H}{2cs^2}$ and $u_{x2} = \dfrac{L}{EA}\left(\dfrac{-P}{1+2c^3}\right)$.

Mathematically, as $\alpha \to 0$, $u_{x1} \to \infty$. This is physically correct because at $\alpha = 0$, all three elements are vertical and cannot withstand any horizontal forces.

Also, as $\alpha \to \frac{pi}{2}$, $u_{x1} \to \infty$. As α increases, two factors play a significant role- the alignment of elements w.r.t. horizontal force H and the length of the elements. Due to the geometry, the length also increases with α. At $\alpha\ \frac{\pi}{2}$, the truss element 1 and 3 are almost horizontal and have infinite length. Although the horizontal alignment of the truss element helps in bearing the force H, the large length results in large displacement ($\because u = \epsilon L$ for uniform stress or uniform strain).

The optimal relation between the two factors here is at

$$\begin{aligned} \alpha_{opt} &= \mathsf{argmin}(1/cs^2) \\ \implies cos\alpha &= 1/2 \\ \implies \alpha &= \frac{\pi}{3} \end{aligned}$$

(d) The axial forces for each element can be obtained by transforming the displacement vector into local axis:

$$\bar{\mathbf{u}}^{(e)} = \begin{bmatrix} c\phi & s\phi & 0 & 0 \\ -s\phi & c\phi & 0 & 0 \\ 0 & 0 & c\phi & s\phi \\ 0 & 0 & -s\phi & c\phi \end{bmatrix} \mathbf{u}^{(e)}$$

with $c = cos\left(\frac{\pi}{2} + \alpha\right)$ and $s = sin\left(\frac{\pi}{2} + \alpha\right)$. Using this relation, we obtain the axial displacement at node 1 as:

$$\begin{aligned} \bar{u}_{x1}^{(1)} &= u_{x1}s - u_{y1}c \\ \implies \quad \bar{u}_{x1}^{(1)} &= \frac{L}{EA}\left(\frac{H}{2cs} + \frac{Pc}{1+2c^3}\right) \\ ||^{\mathsf{rly}} \quad \bar{u}_{x1}^{(2)} &= \frac{L}{EA}\left(\frac{P}{1+2c^3}\right) \\ \bar{u}_{x1}^{(3)} &= \frac{L}{EA}\left(-\frac{H}{2cs} + \frac{Pc}{1+2c^3}\right) \end{aligned}$$

Hence, the axial forces can be calculated as:

$$\begin{aligned} F_a^{(1)} &= \frac{EA}{L/c}(\bar{u}_{x1}^{(1)} - 0) \\ \implies F_a^{(1)} &= \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \\ ||^{\mathsf{rly}} \quad F_a^{(2)} &= \frac{P}{1+2c^3} \\ ||^{\mathsf{rly}} \quad F_a^{(3)} &= -\frac{H}{2s} + \frac{Pc^2}{1+2c^3} \end{aligned}$$

Here we observe that as $\alpha \to 0$, $F_a^{(1)} \to \infty$ and $F_a^{(3)} \to \infty$. Mathematically, the solution is not valid for $\alpha = 0$ because the reduced system of equations in 1b has a singular matrix at $\alpha = 0$. Physically, at $\alpha = 0$, the system of trusses is indeterminate.