UNIVERSITAT POLITÈNCICA DE CATALUNYA

Assignment 1

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February 11, 2019

1 ASSIGNMENT 1.1

Consider the truss problem defined in Figure 1.1. All geometric and material properties: L, α, E and A, as well as the applied forces P and H, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.

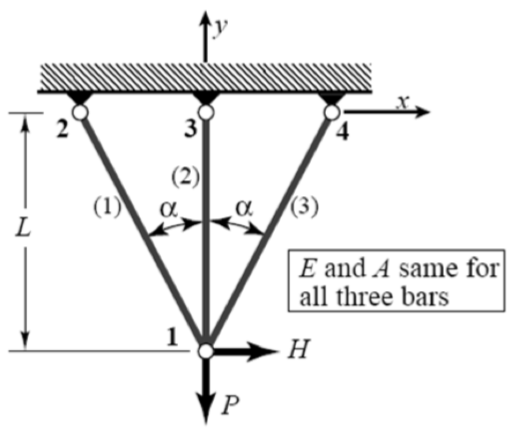

Figure 1.1: Truss structure. Geometry and mechanical features

(a) Show that the master stiffness equations are

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_x1 \\ u_y1 \\ u_x2 \\ u_y2 \\ u_x3 \\ u_y3 \\ u_x4 \\ u_y4 \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.1}$$

in wich $c = cos(\alpha)$ and $s = sin(\alpha)$. Explain from physics why the 5th row and column contain only zeros.

First, the elemental stiffness matrices are calculated using the numbering provided in Figure 1.1. The local numbering is defined in all elements as 1 to the global node 1 and 2 to the global nodes 2, 3 and 4. The local x-axis is defined as going from node 1 to 2, 3 and 4 nodes depending on the element. The local y-axis is defined taking a 90 degrees counter-clockwise rotation of the local x-axis. The elemental stiffness matrices in global form are:

$$\boldsymbol{K}^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c_e^2 & s_e c_e & -c_e^2 & -s_e c_e \\ s_e c_e & s_e^2 & -s_e c_e & -s_e^2 \\ -c_e^2 & -s_e c_e & c_e^2 & s_e c_e \\ -s_e c_e & -s_e^2 & s_e c_e & s_e^2 \end{bmatrix}$$

where s_e and c_e represent the sine and cosine of the angle β of rotation of the local axis over the global axis. β is defined positive if the local axes are rotated counter-clockwise.

From the problem statement and the chose of the local axis explained above it is evident that $\beta_1 = \pi/2 + \alpha$, $\beta_2 = \pi/2$ and $\beta_3 = \pi/2 - \alpha$. It follows that:

$$\begin{aligned}
s_1 &= sin(\beta_1) = sin(\pi/2+\alpha) = cos(\alpha) = c \\
c_1 &= cos(\beta_2) = cos(\pi/2+\alpha) = -sin(\alpha) = -s \\
s_2 &= sin(\beta_2) = sin(\pi/2) = 1 \\
c_2 &= cos(\beta_2) = cos(\pi/2) = 0 \\
s_3 &= sin(\beta_3) = sin(\pi/2-\alpha) = cos(\alpha) = c \\
c_3 &= cos(\beta_3) = cos(\pi/2-\alpha) = sin(\alpha) = s
\end{aligned}$$

Using this data as well as the fact that E and a are constant the elemental matrices are computed:

$$\boldsymbol{K}^1 = \frac{EA}{L/c}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}$$

$$\boldsymbol{K}^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^3 = \frac{EA}{L/c}\begin{bmatrix} s^2 & cs & -s^2 & -cs \\ cs & c^2 & -cs & -c^2 \\ -s^2 & -cs & s^2 & cs \\ -cs & -c^2 & cs & c^2 \end{bmatrix}$$

The assembly of the global matrix is done as follows:

$$\boldsymbol{K} = \frac{EA}{L}\begin{bmatrix} K_{11}^1+K_{11}^2+K_{11}^3 & K_{12}^1+K_{12}^2+K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^2 & K_{14}^2 & K_{13}^3 & K_{14}^3 \\ & K_{22}^1+K_{22}^2+K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^2 & K_{24}^2 & K_{23}^3 & K_{24}^3 \\ & & K_{33}^1 & K_{34}^1 & 0 & 0 & 0 & 0 \\ & & & K_{44}^1 & 0 & 0 & 0 & 0 \\ & & & & K_{33}^2 & K_{34}^2 & 0 & 0 \\ & & & & & K_{44}^2 & 0 & 0 \\ & & & & & & K_{33}^3 & K_{34}^3 \\ & symm & & & & & & K_{44}^3 \end{bmatrix}$$

Substituting the computed values of the elemental stiffness matrices the global stiffness matrix is obtained:

$$\boldsymbol{K} = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ symm & & & & & & & c^3 \end{bmatrix}$$

Which is equal to the global stiffness matrix of Equation 1.1

Finally, to obtain the force vector, it is noted that only two nodal forces are present in the problem. They are applied to node 1 and are H and P in the positive x and negative y directions respectively as follows from the observation of Equation 1.1

(b) Apply the BC's and show the 2-equation modified stiffness system.

Nodes 2, 3 and 4 are restrained with a fixed displacement of **0**. The only non-restrained degrees of freedom are u_{x1} and u_{x2}. That is translated in the master equations eliminating all equations but first and second and all unknowns except u_{x1} and u_{x2}. As the prescribed displacements are **0** there is no contribution on the force vector due to the displacement boundary conditions.
The resulting system is:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

(c) Solve for the displacements u_{x1} and u_{x2}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$?

The resulting system is diagonal, so the solution is straight:

$$u_{x1} = \frac{L}{EA}\frac{H}{2cs^2}$$
$$u_{x2} = \frac{L}{EA}\frac{-P}{1+2c^3}$$

In the limit case when $\alpha \to 0 : c \to 1$ and $s \to 0$. That is translated to $u_{x1} \to \infty$ and $u_{x2} \to \frac{-P}{3}\frac{L}{EA}$.
In this problem, the force H generates a moment around the coordinates origin (node 3). The reactions to that moment are provided by the bars *(1)* and *(3)*. In the limit case as $\alpha \to 0$, the nodes 2, 3 and 4 are closer to each other. That is traduced in reducing the lever arm. For that reason the axial forces $F^{(1)}$ and $F^{(3)}$ become bigger to provide the same reaction moment resulting in a larger deformation. There is even another mechanism that increases the displacement that is that the H force becomes more transversal to the bars as α is reduced meaning that the deflection gets even bigger until infinity for the limit case.
About the other case: $\alpha \to \pi/2$. In this case $c \to 0$ and $s \to 1$. So, $u_{x1} \to \infty$ and $u_{x2} \to \frac{-PL}{EA}$.
In this case, the fact that the solution tends to infinity is for the same reason, the bars *(1)* and *(3)* cannot compensate the moment produced by *H*. This is due to the fact that the length of the to elements tend to infinity and with it, its stiffness is reduced to 0.

(d) Recover the axial forces in the three members. Partial answer: $F^{(3)} = \frac{-H}{2s} + P\frac{c^2}{1+2c^3}$. Why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

Now that the displacements have been solved the reaction forces can be computed as $\overline{\boldsymbol{f}}^e = \overline{\boldsymbol{K}}^e \overline{\boldsymbol{u}}^e = \overline{\boldsymbol{K}}^e \boldsymbol{T}^e \boldsymbol{u}^e$

First element:

$$\frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & c_1 & s_1 \\ 0 & 0 & s_1 & c_1 \end{bmatrix}\begin{bmatrix} u^1_{x1} \\ u^1_{y1} \\ u^1_{x2} \\ u^1_{y2} \end{bmatrix} =$$

$$\frac{EA}{L/c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -s & c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & c & -s \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ \frac{L}{EA}\frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-H}{2s} - P\frac{c^2}{1+2c^3} \\ 0 \\ \frac{H}{2s} + P\frac{c^2}{1+2c^3} \\ 0 \end{bmatrix}$$

$$F^{(1)} = \frac{H}{2s} + P\frac{c^2}{1+2c^3}$$

Second element:

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_2 & s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \\ 0 & 0 & s_2 & c_2 \end{bmatrix}\begin{bmatrix} u^2_{x1} \\ u^2_{y1} \\ u^2_{x2} \\ u^2_{y2} \end{bmatrix} =$$

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{L}{EA}\frac{H}{2cs^2} \\ \frac{L}{EA}\frac{-P}{1+2c^3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-P}{1+2c^3} \\ 0 \\ \frac{P}{1+2c^3} \\ 0 \end{bmatrix}$$

$$F^{(2)} = \frac{P}{1+2c^3}$$

Third element:

$$\frac{EA}{L/c}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}c_3 & s_3 & 0 & 0\\ s_3 & c_3 & 0 & 0\\ 0 & 0 & c_3 & s_3\\ 0 & 0 & s_3 & c_3\end{bmatrix}\begin{bmatrix}u_{x1}^3\\ u_{y1}^3\\ u_{x2}^3\\ u_{y2}^3\end{bmatrix}=$$

$$\frac{EA}{L/c}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}s & c & 0 & 0\\ c & s & 0 & 0\\ 0 & 0 & s & c\\ 0 & 0 & c & s\end{bmatrix}\begin{bmatrix}\frac{L}{EA}\frac{H}{2cs^2}\\ \frac{L}{EA}\frac{-P}{1+2c^3}\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}\frac{H}{2s}-P\frac{c^2}{1+2c^3}\\ 0\\ \frac{-H}{2s}+P\frac{c^2}{1+2c^3}\\ 0\end{bmatrix}$$

$$F^{(3)}=\frac{-H}{2s}+P\frac{c^2}{1+2c^3}$$

2 Assignment 1.2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

The elemental matrices of elements (1) and (2) used in this problem are the same than in the example. The matrices (3) and (4) are just the double of (3) in the example as the elements are equal but half length. So the global stiffness matrix is:

$$\boldsymbol{K}=\begin{bmatrix}K_{11}^1+K_{11}^3 & K_{12}^1+K_{12}^3 & K_{13}^1 & K_{14}^1 & 0 & 0 & K_{13}^3 & K_{14}^3\\ & K_{22}^1+K_{22}^3 & K_{23}^1 & K_{24}^1 & 0 & 0 & K_{23}^3 & K_{24}^3\\ & & K_{33}^1+K_{11}^2 & K_{34}^1+K_{12}^2 & K_{13}^2 & K_{14}^2 & 0 & 0\\ & & & K_{44}^1+K_{22}^2 & K_{23}^2 & K_{24}^2 & 0 & 0\\ & & & & K_{33}^2+K_{11}^4 & K_{34}^2+K_{12}^4 & K_{13}^4 & K_{14}^4\\ & & & & & K_{44}^2+K_{22}^4 & K_{23}^4 & K_{24}^4\\ & & & & & & K_{33}^3+K_{33}^4 & K_{34}^3+K_{34}^4\\ symm & & & & & & & K_{44}^3+K_{44}^4\end{bmatrix}$$

As the degrees of freedom of the node 1 and the y-displacement of the node 2 are restricted, the first, second and fourth rows and columns have to be removed from the global stiffness matrix:

$$\hat{\boldsymbol{K}}=\begin{bmatrix}K_{33}^1+K_{11}^2 & K_{13}^2 & K_{14}^2 & 0 & 0\\ & K_{33}^2+K_{11}^4 & K_{34}^2+K_{12}^4 & K_{13}^4 & K_{14}^4\\ & & K_{44}^2+K_{22}^4 & K_{23}^4 & K_{24}^4\\ & & & K_{33}^3+K_{33}^4 & K_{34}^3+K_{34}^4\\ symm & & & & K_{44}^3+K_{44}^4\end{bmatrix}=\begin{bmatrix}10 & 0 & 0 & 0 & 0\\ 0 & 5 & 5 & -5 & -5\\ 0 & 5 & 10 & -5 & -5\\ 0 & -5 & -5 & 10 & 10\\ 0 & -5 & -5 & 10 & 10\end{bmatrix}$$

It is obvious that this matrix is singular meaning that the problem is not well posed. This is because the system is under-constrained. It is well-known that bar elements can only be used in fully triangulated trusses as they are unable to resist shear forces nor bending moments. That is the reason why the matrix is singular: the structure is unstable.