Primal and Mixed formulations of the linear elasticity equation

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1 Introduction

The internship was aimed to compare different approximation techniques for a classical problem in computational mechanics. We Considered the linear elasticity problem under the assumption of small deformations and small displacements.

Under the assumption of small deformations and small displacements the strain tensor is approximated as:

$$\epsilon(u) = e(u) := \frac{1}{2}(\nabla u + \nabla u^T) \tag{1}$$

The relationship between stress tensor and and strain tensor is known as:

$$\sigma = Ae(u) = 2\mu e(u) + \lambda tr(e(u))I_d \tag{2}$$

where the Lame contstants μ, λ are defined as:

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu := \frac{E}{2(1+\nu)}$$

Also we remark that the elasticity tensor exist and is invertible as long as $\nu < \frac{1}{2}$ or equivalently $\lambda < \infty$. Within this framework we introduce the compliance tensor A^{-1} by it's application to stress tensor we can compute the strain tensor:

$$e(u) = A^{-1}\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{2\mu(d\lambda + 2\mu)}tr(\sigma)I_d \tag{3}$$

Let $\Omega \subset \mathbb{R}^d$, d=2,3 be an open connected domain representing the body under analysis. The governing equations for the problem of linear elasticity set in an open connected domain Ω. Let the total boundary of the domain be defined $\partial\Omega = \Gamma^N \cup \Gamma^D$ i.e boundary is split into Neumann and Dirichlet boundary.

$$\begin{aligned}
-\nabla \cdot \sigma = \mathbf{f} \quad & in \;\; \Omega \\
\sigma = A \cdot e(u) \quad & in \;\; \Omega \\
\sigma n = g \qquad\quad & on \;\; \Gamma^N \\
u = 0 \qquad\quad & on \;\; \Gamma^D
\end{aligned} \tag{4}$$

2 Pure Displacement Variational formulation (Primal Formulation)

In pure displacement formulation we express the stress tensor σ in terms of displacement $\mathbf{u}$ using (2) and we seek the displacement field within the Sobolev space and we define the space as:

$$V := \mathbf{H^{-1}_{0,\Gamma^D}}(\Omega; R^d) = (\mathbf{v} \in H^{-1}(\Omega; R^d : \mathbf{v}) = 0 \;\; on \;\; \Gamma^D$$

$$a(\mathbf{u}, \delta\mathbf{u}) = F(\delta u) \quad \forall \delta u \in V$$

where the bilinear form is defined:

$$a(u, \delta u) := \int_\Omega Ae(u) : e(\delta u) dx \quad , \quad F(\delta u) := \int_\Omega \mathbf{f} \cdot \delta\mathbf{u} dx + \int_{\Gamma^N} \mathbf{g} \cdot \delta\mathbf{u} ds$$

The above mentioned is the variational formulation of the displacement formulation.

3 Mixed Variational Formulations

A major drawback of the pure displacement variational formulation is the indirect evaluation of the stress tensor which is not computed as part of the solution of the linear elasticity problem but may only be derived from (2) via a post-processing of the displacement field $\mathbf{u}$. This issue was resolved by mixed variational formulations in which the target solution is the pair $(\sigma, \mathbf{u})$

representing respectively the stress and displacement fields. This family of approaches was first proposed by Reissner and these methods are summarize as Hellinger-Reissner methods. The stress tensor is sought in a subspace of $\mathbb{H}(div, \Omega; \mathbb{S}_d)$, in particular stress belong to the space of square-integrable tensors whose row-wise divergence is square-integrable strongly enforces the conservation of momentum. Moreover, the symmetry of the stress tensor is a simplified way of expressing the conservation of angular momentum for the system under analysis. It is well-known that imposing exactly a conservation law is not trivial. Hence, strongly enforcing a second conservation law by requiring the stress tensor to be symmetric is likely to be difficult. Let $\mathbb{M}_d$ be the space of dd matrices and $\mathbb{K}_d$ be the space of dd skew-symmetric matrices. We define the spaces V:=L^2 $(\Omega; \mathbb{R}_d)$, Q:=$L^2(\Omega; \mathbb{K}_d), \Sigma := [\tau \in H(div, \Omega; \mathbb{M}_d) : \tau n = gon\Gamma^N]$ and $\Sigma_0 := [\tau \in H(div, \Omega; \mathbb{M}_d) : \tau n = 0on\Gamma^N]$ morpver we introduce the space W:=VxQ and we seek the solution such that $(\sigma, \delta\sigma) \in \Sigma x W$

$$a(\sigma, \delta\sigma) + b(\delta\sigma, (u, \nu)) = 0 \qquad \forall \delta\sigma \in \Sigma_0$$

$$b(\sigma, (\delta u, \delta\nu)) = F(\delta u) \qquad \forall (\delta u, \delta\nu) \in W \tag{5}$$

The bilinear and linear forms have the following expressions:

$$a(\sigma, \delta\sigma) := \int_\Omega A^{-1}\sigma : \delta\sigma dx \;\; , \;\; b(\sigma, (\delta u, \delta\eta)) := \int_\Omega (\nabla \cdot \sigma) \cdot \delta u dx + \frac{1}{2\mu} \int_\Omega \sigma : \delta\eta dx$$

$$F(\delta u) := -\int_\Omega \mathbf{f} \cdot \delta u dx \tag{6}$$

4 Results

Freefem was used in order to solve the linear elasticity equation during the Internship. FreeFem++ is a partial differential equation solver. It has its own language. freefem scripts can solve multiphysics non linear systems in 2D and 3D. FreeFem++ is written in C++ and the FreeFem++ language is a C++ idiom.

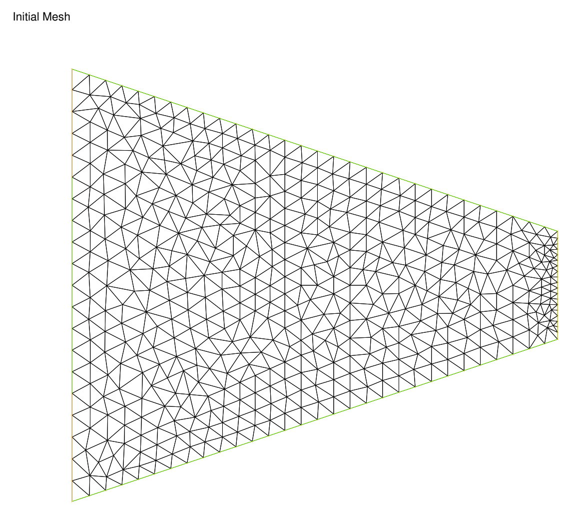
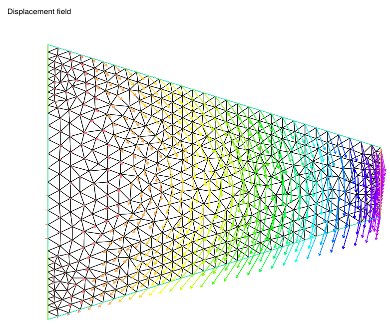


Figure 1: Geometry and meshed Domain

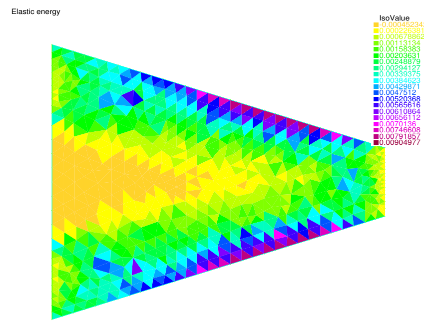
The domain was chosen as that to be of a tapered beam and the whole domain was meshed with freefem using triangular elements see Figure 1. The geometry consist of 3 boundaries, Dirichlet, Neumann and Free. The left side of beam is Dirichlet, left side of beam is fixed since we are considering the case of a cantilever beam. Right side of the beam is Neumann and top and bottom is Free boundary.

Freefem scripts were written and results were analyzed for different test cases that are shown in the below.

It can be observed that in case 3 since the Poisson's ratio increases and exceeds the limit $-1 < /nu < \frac{1}{2}$ thus the elastic tensor is no longer invertible and we obtain an exploded and obsolete result see Figure 4. The case 2 is of when there is load applied to the right side of the beam. Case 1 is for when there is no load applied on the beam, i.e. free cantilever beam under action of no load.

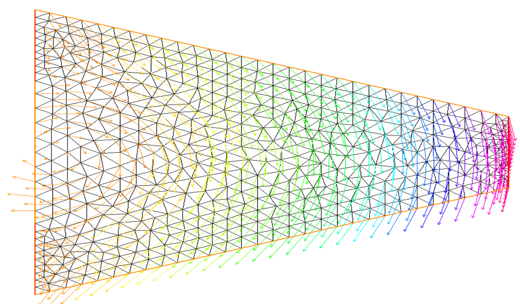


(a) Displacement vectors

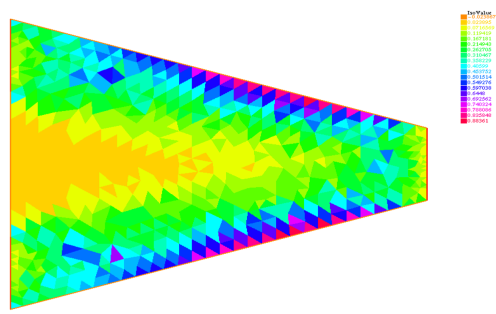


(b) Elastic Energy

Figure 2: Case 1: No load on Beam

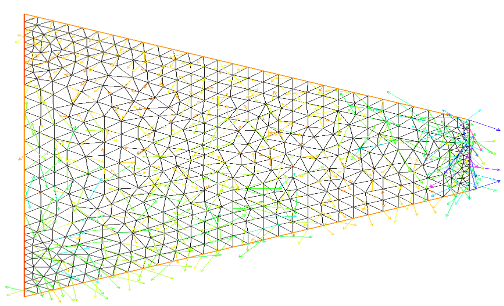


(a) Displacement vectors

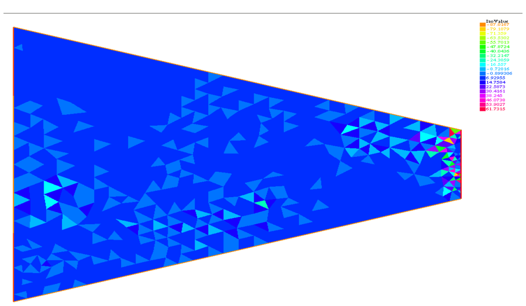


(b) Elastic Energy

Figure 3: Case 2: Load applied on right side of beam



(a) Displacement vectors



(b) Elastic Energy

Figure 4: Case 3: For $\nu > \frac{1}{2}$