UPC - BARCELONA TECH
MSc Computational Mechanics
Spring 2018

Finite Elements in Fluid

Lab 1 1D Steady Convection-Diffusion

Due 13/02/2018

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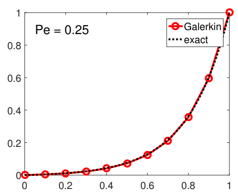
1 Problem Statement

1D convection-diffusion equation with constant coefficients and Dirichlet boundary conditions:
$au_x - \nu u_{xx} = sx \in [0, 1]$
$u(0) = u_0; u(1) = u_1$
The script main allows us to solve three different examples,as follows

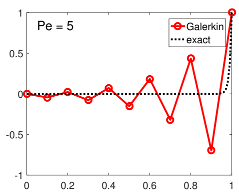
1. $s = 0, u_0 = 0, u_1 = 1$
2. $s = 1.u_0 = 0, u_1 = 0$
3. $s = sin(\pi x), u_0 = 0, u1 = 1$

2 Solving problem 1 using Galerkin's method

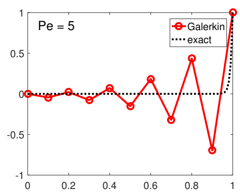
The first problem is solved using Galerkin's method for various convection and diffusion coefficients and number of elements are varied to observe the behaviour of the Galerkin's method.



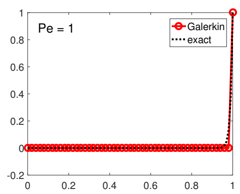
Case 1



Case 2



Case 3



Case 4

Case 1 $a = 1, \nu = 0.2, 10$ Linear elements	Case 2 $a = 20, \nu = 0.2, 10$ Linear elements
Case 3 $a = 1, \nu = 0.01, 10$ Linear elements	Case 4 $a = 1, \nu = 0.01, 50$ Linear elements

It is Known that Galerkin Method is good to solve convection-Diffusion problems. But it is also known that they are not suitable for solving convection dominated problem. To have an equal effects of convection as well as diffusion in a problem, Peclet number is introduced.
It can be observed that as Peclet No. increases Galerkin method's approximate solution wavers off from the exact solution. As it is displayed in the plot 2 and 3 above. And moreover it oscillates along the x axis. Also, it can be observed that the plots for case 2 and case 3, graph is identical. This is because the ratio $\frac{a}{\nu}$ remains constant. Thus, the Pe is same for both case, hence the Galerkin approximate solution is same for both. In the fourth plot it can be seen that the Galerkin solution matches the exact solution. This is because of the space width h. For any method it is known that as number of elements increase (mesh is refined) the solution is improved as h decreases.

3 Comparison between SU, SUPG and GLS

To compare the behavior of the three methods mentioned above, the third problem(main.m) is evaluated for the third case mentioned above. Also, the Matlab code for SUPG and GLS are completed according to the ppt (Steady 1D convection Examples) slide no.30 and 31. The line of code edited are as follows:

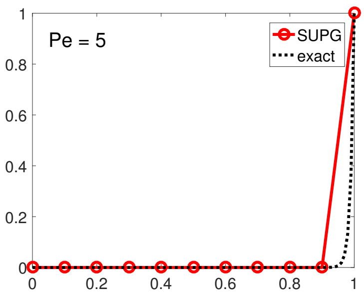
```
Nx_ig   = Nxi(ig,:)*2/h;
N2x_ig = N2xi(ig,:)*4/h^2;
w_ig = wgp(ig)*h/2;
Ke = Ke + w_ig*(N_ig'*(a*Nx_ig) + Nx_ig'*(nu*Nx_ig)) + ...
    + tau*w_ig*( (a*Nx_ig)'*(a*Nx_ig)) - nu*N2x_ig));
x = N_ig*Xe; % x-coordinate of the gauss point
s = SourceTerm(x,example);
fe = fe + w_ig*(N_ig')*s + tau*w_ig*(a*Nx_ig)'*s;
```

Change done to obtain SUPG code

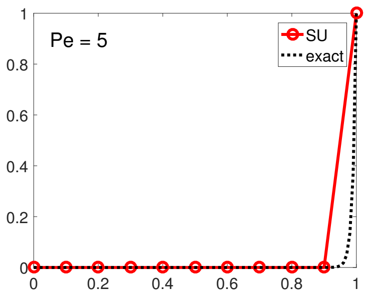
```
Nx_ig   = Nxi(ig,:)*2/h;
N2x_ig = N2xi(ig,:)*4/h^2;
w_ig = wgp(ig)*h/2;
Ke = Ke + w_ig*(N_ig'*(a*Nx_ig) + Nx_ig'*(nu*Nx_ig))   ...
    + tau*w_ig*( (a*Nx_ig - nu*N2x_ig)'*(a*Nx_ig - nu*N2x_ig));
x = N_ig*Xe; % x-coordinate of the gauss point
s = SourceTerm(x,example);
fe = fe + w_ig*(N_ig')*s + tau*w_ig*(a*Nx_ig - nu*N2x_ig)'*s;
```

Change done to obtain GLS code

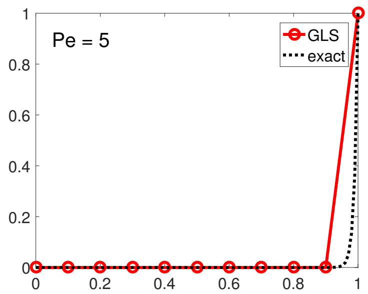
Also, please note that N2_ig is initiated earlier in the code outside the loop as `N2_xi = referenceElement.N2_xi;`. The above methods are also used to solve problem 1 stated above, using parameters of case 3. But it is clear that the differences between the solutions obtained by different methods are not apparent. Hence, it is better to use problem 3 as an example to observe differences in behaviour of individual methods.



Solution by SUPG method

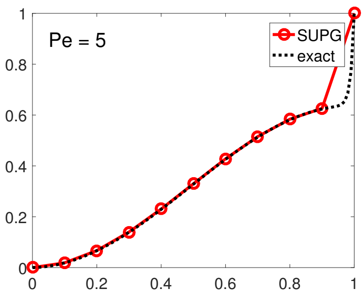


Solution by SU method

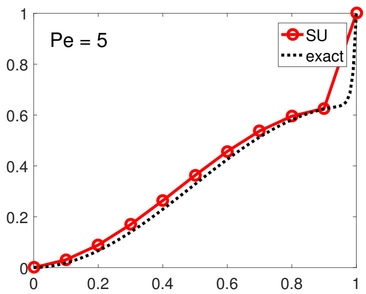


Solution by GLS method

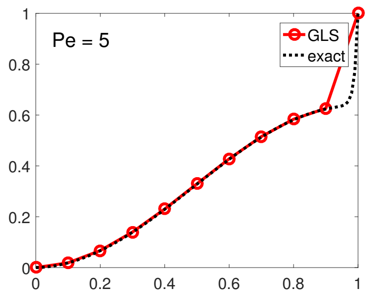
The three methods are used to get a solution for the problem 3 stated in the problem statement section. The following graphs are plotted for the approximate and exact solution for each method for case 3.



Solution by SUPG method



Solution by SU method



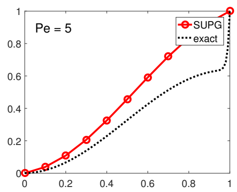
Solution by GLS method

It is apparent from the plots that the solution given by SUPG and GLS is identical. This is because the problem stated is 1D, steady state, and also it is given that $\sigma = 0$ in the problem. Hence the terms

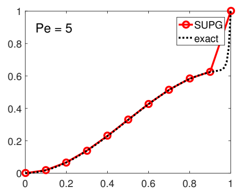
added for stabilization for SUPG and GLS, after cancelling out remains same, as seen in the code line 45 for both methods. Note that the term N2x_ig will vanish as it is the 2^{nd} order derivative. Since we are using 1D Linear elements.
The SU solution does not match with the exact solution because it over compensates(soomthens the solution more than needed). It can be said that diffusion term is dominating. Also, in SU the residual term is not considered, which is considered in SUPG and GLS, hence they provide a better solution.
For SUPG and GLS it can be seen that the approximate solution matches with the exact nodal solution, until it over shoots. The approximate solution of SU does match with exact nodal solution, but it does for SUPG and GLS because they are improved using stabilization methods.

4 Behaviour of SUPG by varying Stabilization Parameter τ

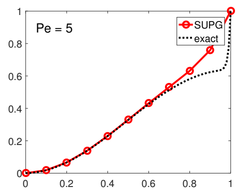
The third problem is solved using the SUPG method. The optimum stabilization parameter is $\tau = 0.4005$. But instead we use $\tau = 1$ and $\tau = 0.1$ (with other parameters as case 3 mentioned earlier).



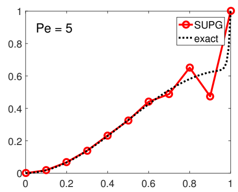
Solution by $\tau = 1$



Solution by $\tau = 0.4005$



Solution by $\tau = 0.1$



Solution by $\tau = 0.01$

It is a fact that the added stabilization parameter is unsymmetrical. As the value of τ increases above the optimal value the approximate solution is far away from the exact solution. This is because when the stabilisation parameter is increased from the optimal value, the method over compensates. When the value of τ decreases from optimal value, it is close to the nodal solution. But as we decrease the value of τ more and more ($\tau \to 0$), this methods behaves as Galerkin method(solution oscillates between nodes). And it is known that $\tau = 0$ yields Galerkin solution.