FEF midterm 2020

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Ex 1

Inviscid Burgers - $u_t + u u_x = 0 \quad x \in (0,1) \quad t > 0$

$u = u_0(x) \quad x \in [0,1] \quad t = 0$

$u = 1 \quad x = 0 \quad t > 0$

$$u_0(x) = \begin{cases} 1 & x \in [0,p) \\ 1 - \dfrac{x-p}{q-p} & x \in [p,q] \\ 0 & x \in (q,1] \end{cases}$$

$p = 0.64$

$q = 0.84$

Spatial discretization of ~~inviscid burgers~~ of TG-2 approx. is-

$$\left(w, \frac{\Delta u}{\Delta t}\right) = -\left(w, a\cdot\nabla u^n - \frac{\Delta t}{2}(a\cdot\nabla)^2 u^n\right) + \left(w, s^n + \frac{\Delta t}{2}\left(s_t^n - a\cdot\nabla s^n\right)\right)$$

Integrating this by parts becomes,

$$\left(w, \frac{\Delta u}{\Delta t}\right) = \left(a\cdot\nabla w, u^n + \frac{\Delta t}{2}[s^n - (a\cdot\nabla)u^n]\right) - \left((a\cdot n)w, u^n + \frac{\Delta t}{2}[s^n - (a\cdot\nabla)u^n]\right)_{\Gamma_{out}} + \left(w, h^{n+1/2}\right)_{\Gamma_N^{in}} + \left(w, s^n + \frac{\Delta t}{2} s_t^n\right)$$

a)

TG2 approx $\rightarrow$ $\dfrac{u^{n+1} - u^n}{\Delta t} = u_t^n + \dfrac{\Delta t}{2} u_{tt}^n$

here, $u_t = -f_x$

$u_{tt} = -f_{xt} = -f_{tx} = -(f_u u_t)_x = (u^2 u_x)_x.$

Burgers equ. becomes $\rightarrow$ $\dfrac{u^{n+1} - u^n}{\Delta t} = -f_x^n + \dfrac{\Delta t}{2}\left((u^n)^2 u_x^n\right)_x$

After integration by parts,

$$\int_0^L w \frac{u^{n+1} - u^n}{\Delta t} dx = \int_0^L w_x f^n dx - \frac{\Delta t}{2} \int_0^L w_x (u^n)^2 u_x^n \, dx \quad - \left[w\left(f^n - \frac{\Delta t}{2} (u^n)^2 u_x^n\right) \right]_{x=0}^{x=L}$$

The boundary term (last term on RHS) can be written as -

$$= -\left[w\left(f^n - \frac{\Delta t}{2} (u^n)^2 u_x^n\right) \right]_{x=0}^{x=L}$$

$$= -\left[w\left(f^n + \frac{\Delta t}{2} f_t^n\right) \right]_{x=0}^{x=L}$$

f is element-wise const· representation.

c)

Alternative choices to represent fluxes (non-linear) are -

1. Classical Representation -

f is determined from 'u' at 2 element gauss points & a two-point Gaussian quadrature is used to evaluate convective term.

2. Group representation -

f is linearly interpolated using its evaluation at the element nodes.

Ex. 2

$$-\nu u_{xx} + \beta u_x = 0 \qquad x \in (-1, 1)$$

$u = 0 \qquad x = -1 \qquad \nu = 0.03$

$u = -1 \qquad x = 1 \qquad \beta = 1.8$

a) $\beta u_x - \nu u_{xx} = 0.$

The weak form of this equ. is -

$$\int_{-1}^{1} (w \cdot \beta u_x + w_x \cdot \nu u_x)\, dx = 0$$

Shape func. are $N_1(\xi) = \frac{1}{2}(1-\xi)$, $N_2(\xi) = \frac{1}{2}(1+\xi)$

Discrete equ. at nodes takes the form

$$\int_{-1}^{1} \Sigma \left(\beta N_A \frac{\partial N_B}{\partial x} + \nu \frac{\partial N_A}{\partial x} \frac{\partial N_B}{\partial x} \right) u_B \, dx - 0$$

The convection & diffusion matrices C^e & K^e resp. are -

$$C^e = \beta \int_{\Omega^e} \begin{bmatrix} N_A \frac{\partial N_A}{\partial x} & N_A \frac{\partial N_B}{\partial x} \\ N_B \frac{\partial N_A}{\partial x} & N_B \frac{\partial N_B}{\partial x} \end{bmatrix} dx$$

$$K^e = \nu \int_{\Omega^e} \begin{bmatrix} \frac{\partial N_A}{\partial x} \frac{\partial N_A}{\partial x} & \frac{\partial N_A}{\partial x} \frac{\partial N_B}{\partial x} \\ \frac{\partial N_B}{\partial x} \frac{\partial N_A}{\partial x} & \frac{\partial N_B}{\partial x} \frac{\partial N_B}{\partial x} \end{bmatrix} dx$$

Discrete form of Galerkin finite element formulation -

$$\beta \left(\frac{u_{j+1} - u_{j-1}}{2h} \right) - \nu \left(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \right) = 0.$$

b)

SGS method equ -

$$\int w(\beta\cdot\nabla u)\,dx + \int \nabla w\cdot(\nu\nabla u)\,dx + \sum_e \int P(w)\,\tau\,R(u)\,dx = 0$$

$$\sum_e \int \left(\beta\cdot\nabla u + \nabla\cdot(\nu\nabla u)\right)\tau\left(\beta\cdot\nabla u - \nabla\cdot(\nu\nabla u)\right)dx$$

stabilization term.

Galerkin method is unstable for $Pe > 1$. But it gives good results for $Pe < 1$.

SGS method overcomes the instabilities introduced by GLS method which is supposed to be more stable for $Pe > 1$.

Pe in this question is > 1. ∴ Galerkin will show oscillations.

$Pe = \frac{\beta h}{2\nu}$.

SGS method will provide smooth results at $Pe > 1$.

c)

Consistent stabilization means diffusion is added where the mesh if finer. & smooth the sol.

Consistent stabilization gives less numerical diffusion when the numerical sol. appears more close to exact sol.

The stabilization parameter should ~~decrease~~ vanish as mesh is refined.

Stabilization parameter is influenced by,

diffusivity, mesh size, convection vel., reaction cof (if present)