Finite Elements in Fluids

HDG assignment 18

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1 Problem statement

Consider the domain $\Omega = [0,1]^2$ such that $\delta\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ with

1. $\Gamma_N := \{(x,y) \in \mathbb{R}^2 : y = 0\}$,
2. $\Gamma_R := \{(x,y) \in \mathbb{R}^2 : y = 1\}$,
3. $\Gamma_N := \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ and } x = 1\}$.

The following second-order linear scalar partial differential equation is defined

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = s & \text{in } \Omega \\ u = u_D & \text{on } \Gamma_D \\ \mathbf{n} \cdot (\kappa \nabla u) = t & \text{on } \Gamma_N \\ \mathbf{n} \cdot (\kappa \nabla u) + \gamma u = g & \text{on } \Gamma_R \end{cases}$$

where κ and γ are the diffusion and convection coefficients, repectively, $\mathbf{n}$ is the outward unit normal vector to the boundary, s is a volumetric source term and u_D, t and g are the Dirichlet, Neumann and Robin data imposed on the corresponding portions of the boundary $\delta\Omega$.

2 HDG formulation of the problem

STRONG FORM

We consider $\Omega = \cup_{i=1}^{n_{el}} \Omega_e$ a partition of the domain and $\Gamma = [\cup_{i=1}^{n_{el}} \delta\Omega_e] - \delta\Omega$. Thus we can write the strong form of the broken computational domain as

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = s & \text{in } \Omega_i \quad \text{for } i = 1, ..., n_{el} \\ u = u_D & \text{on } \Gamma_D \\ \mathbf{n} \cdot (\kappa \nabla u) = t & \text{on } \Gamma_N \\ \mathbf{n} \cdot (\kappa \nabla u) + \gamma u = g & \text{on } \Gamma_R \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma \\ [\![\nabla u\mathbf{n}]\!] = 0 & \text{on } \Gamma \end{cases}$$

And introducing a mixed variable $\mathbf{q} = -\kappa \nabla u$ the system can be written as

$$\begin{cases} \nabla \cdot \mathbf{q} = s & \text{in } \Omega_i \quad \text{for } i = 1, ..., n_{el} \\ \mathbf{q} + \kappa \nabla u = 0 & \text{in } \Omega_i \quad \text{for } i = 1, ..., n_{el} \\ u = u_D & \text{on } \Gamma_D \\ \mathbf{n} \cdot \mathbf{q} = -t & \text{on } \Gamma_N \\ -\mathbf{n} \cdot \mathbf{q} + \gamma u = g & \text{on } \Gamma_R \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma \\ [\![\mathbf{q} \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma \end{cases}$$

Thus, we can state the local problem as, find q_i and u_i for each $i = 1, ..., n_{el}$

$$\begin{cases} \nabla \cdot \mathbf{q_i} = s & \text{in } \Omega_i \\ \mathbf{q_i} + \kappa \nabla u_i = 0 & \text{in } \Omega_i \\ u = u_D & \text{on } \delta\Omega_i \cap \Gamma_D \\ u = \hat{u} & \text{on } \delta\Omega_i - \Gamma_D \end{cases}$$

for $\hat{u} \in \mathcal{L}_2(\Gamma \cup \Gamma_N)$ given. Thus, we will have to solve the global problem for the mixed variable $\hat{u}$

$$\begin{cases} [\![\mathbf{q} \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma \\ \mathbf{n} \cdot \mathbf{q} = -t & \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{q} = \gamma \hat{u} - g & \text{on } \Gamma_R \end{cases}$$

WEAK FORM:

Considering ω in the same space as $\mathbf{q_i}$ ($[\mathcal{H}^1(\Omega)]^{n_{sd}}$) and v in the same space as u ($\mathcal{H}^1(\Omega)$), and using the divergence theorem we end with

$$\begin{aligned} -\int_{\Omega_i} \nabla v \cdot \mathbf{q_i} d\Omega + \int_{\delta\Omega_i} v\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} d\Gamma &= \int_{\Omega_i} vsd\Omega \\ \int_{\Omega_i} \omega\cdot\mathbf{q_i} d\Omega - \int_{\Omega_i} \nabla\cdot\omega\kappa u d\Omega &= -\int_{\delta\Omega_i\cap\Gamma_D} \mathbf{n}\cdot\omega\kappa u_D d\Gamma - \int_{\delta\Omega_i - \Gamma_D} \mathbf{n}\cdot\omega\kappa\hat{u} d\Gamma \end{aligned}$$

where the flux $\hat{\mathbf{q}}_\mathbf{i}$ is defined as

$$\hat{\mathbf{q}}_\mathbf{i} = \begin{cases} \mathbf{n}\cdot\mathbf{q_i} + \tau_i(u_i - u_D) & \text{on } \delta\Omega_i\cap\Gamma_D \\ \mathbf{n}\cdot\mathbf{q_i} + \tau_i(u_i - \hat{u}) & \text{on } \delta\Omega_i - \Gamma_D \end{cases}$$

being τ_i a local stabilization parameter.

And subtituting in it we get

$$\begin{aligned} -\int_{\Omega_i} \nabla v\cdot\mathbf{q_i} d\Omega + \int_{\delta\Omega_i} v\tau_i u_i d\Gamma + \int_{\delta\Omega_i} v\mathbf{n}\cdot\mathbf{q_i} d\Gamma &= \int_{\Omega_i} vsd\Omega + \int_{\delta\Omega_i\cap\Gamma_D} v\tau_i u_D d\Gamma + \int_{\delta\Omega_i-\Gamma_D} v\tau_i\hat{u} d\Gamma \\ \int_{\Omega_i} \omega\cdot\mathbf{q_i} d\Omega - \int_{\Omega_i} \nabla\cdot\omega\kappa u d\Omega &= -\int_{\delta\Omega_i\cap\Gamma_D} \mathbf{n}\cdot\omega\kappa u_D d\Gamma - \int_{\delta\Omega_i-\Gamma_D} \mathbf{n}\cdot\omega\kappa\hat{u} d\Gamma \end{aligned}$$

Since $\delta\Omega_i = (\delta\Omega_i\cap\Gamma_D)\cup(\delta\Omega_i\cap\Gamma_N)\cup(\delta\Omega_i\cap\Gamma_R)\cup(\delta\cap\Gamma)$, the weak form of the local problem is: find (q_i, u_i) such that

$$\begin{aligned} & - \int_{\Omega_i} \nabla v\cdot\mathbf{q_i} d\Omega + \int_{\delta\Omega_i} v\tau_i u_i d\Gamma \\ & = \int_{\Omega_i} vsd\Omega + \int_{\delta\Omega_i\cap\Gamma_D} v\tau_i u_D d\Gamma + \int_{\delta\Omega_i-\Gamma_D} v\tau_i\hat{u} d\Gamma + \int_{\delta\Omega_i\cap\Gamma_N} vtd\Gamma - \int_{\delta\Omega_i\cap\Gamma_R} v(\gamma\hat{u} - g) d\Gamma \\ & \int_{\Omega_i} \omega\cdot\mathbf{q_i} d\Omega - \int_{\Omega_i} \nabla\cdot\omega\kappa u d\Omega = -\int_{\delta\Omega_i\cap\Gamma_D} \mathbf{n}\cdot\omega\kappa u_D d\Gamma - \int_{\delta\Omega_i-\Gamma_D} \mathbf{n}\cdot\omega\kappa\hat{u} d\Gamma \end{aligned}$$

for all v and ω.

Considering now the **global problem** and integrating along all the partitions

$$\sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i-\delta\Omega} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} d\Gamma \right\} + \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} + td\Gamma \right\} + \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} + g - \gamma\hat{u} d\Gamma \right\} = 0$$

where $\hat{v}$ belongs to the same space as $\hat{u}$, that is $\mathcal{V}^h(\Gamma\cup\Gamma_N\cup\Gamma_R)$.

Using the expression of $\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i}$ in each integral,

$$\begin{aligned} \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i-\delta\Omega} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} d\Gamma \right\} &= \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i-\delta\Omega} \hat{v}\mathbf{n}\cdot\mathbf{q_i} d\Gamma + \int_{\delta\Omega_i-\delta\Omega} \hat{v}\tau_i u_i d\Gamma - \int_{\delta\Omega_i-\delta\Omega} \hat{v}\tau_i\hat{u} d\Gamma \right\} \\ \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} + td\Gamma \right\} &= \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}\mathbf{n}\cdot\mathbf{q_i} d\Gamma + \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}\tau_i u_i d\Gamma - \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}\tau_i\hat{u} d\Gamma + \int_{\delta\Omega_i\cap\Gamma_N} \hat{v}td\Gamma \right\} \\ \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}\mathbf{n}\cdot\hat{\mathbf{q}}_\mathbf{i} + g - \gamma\hat{u} d\Gamma \right\} &= \sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}\mathbf{n}\cdot\mathbf{q_i} d\Gamma + \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}\tau_i u_i d\Gamma - \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}(\tau_i + \gamma)\hat{u} d\Gamma + \int_{\delta\Omega_i\cap\Gamma_R} \hat{v}gd\Gamma \right\} \end{aligned}$$

Thus, we end with: find $\hat{u}$ such that

$$\sum_{i=1}^{n_{el}} \left\{ \int_{\delta\Omega_i - \Gamma_D} \hat{v}\mathbf{n}\cdot\mathbf{q_i} d\Gamma + \int_{\delta\Omega_i - \Gamma_D} \hat{v}\tau_i u_i d\Gamma - \int_{\delta\Omega_i - \Gamma_D} \hat{v}\tau_i \hat{u} d\Gamma \right\} = \sum_{i=1}^{n_{el}} \left\{ -\int_{\delta\Omega_i \cap \Gamma_N} \hat{v} t d\Gamma - \int_{\delta\Omega_i \cap \Gamma_R} \hat{v} g d\Gamma + \int_{\delta\Omega_i \cap \Gamma_R} \hat{v}\gamma\hat{u} d\Gamma \right\}$$

for all $\hat{v}$.