Homework 10

Navier-Stokes numerical example

Problem formulation

Unsteady Navier-Stokes problem:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \boldsymbol{\nabla}p = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{v}_D & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{t} & \text{on } \Gamma_N \end{cases}$$

In order to solve an unsteady Navier-Stokes problem a proper time integration scheme must be chosen. The time integration can be performed using a monolithic approach or so-called fractional-step/projection methods. This report deals with the implementation of one representative for each approach, namely:

- **Semi-implicit first order monolitic scheme**
- **Chorin-Temam projection method.**

Discretization of the semi-implicit first order monolithic scheme

The momentum conservation equation can be rewritten as:

$$\boldsymbol{v}_t + \mathcal{C}(\boldsymbol{v}) + \mathcal{K}(\boldsymbol{v}) + \boldsymbol{\nabla}p = \boldsymbol{f},$$

where

$$\mathcal{C}(\boldsymbol{v}) = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$$
$$\mathcal{K}(\boldsymbol{v}) = -\nu\nabla^2\boldsymbol{v}.$$

By applying a classical one-step integration scheme the equation takes the form:

$$\begin{aligned} &\frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^n}{\Delta t} + \mathcal{C}(\boldsymbol{v}) + \mathcal{K}(\boldsymbol{v}^{n+\theta}) + \boldsymbol{\nabla}p^{n+\theta} = \boldsymbol{f}^{n+\theta} \\ &\boldsymbol{\nabla}\cdot\boldsymbol{v}^{n+\theta} = 0 \\ \text{with}\quad &\Delta t = t^{n+1} - t^n \qquad f^n \approx f(t^n) \\ &f^{n+\theta} = \theta f^{n+1} + (1-\theta)f^n. \end{aligned}$$

Since there is no time derivative for p, the pressure gradient can be simply evaluated at time t^{n+1}. The same holds for the incompressibility constraint as long as the initial condition is divergence-free and because $f^{n+\theta} = \theta f^{n+1} + (1-\theta)f^n$ holds for any function.

Choosing a semi-explicit approximation for the convective term $\mathcal{C}(\boldsymbol{v})$ we finally obtain the first order approximation as follows:

$$\begin{aligned}&\frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^{n}}{\Delta t}+\mathcal{C}(\boldsymbol{v})+\mathcal{K}(\boldsymbol{v}^{n+1})+\boldsymbol{\nabla}p^{n+1}=\boldsymbol{f}^{n+1}\\&\boldsymbol{\nabla}\cdot\boldsymbol{v}^{n+1}=0,\end{aligned}$$

where

$$\mathcal{C}(\boldsymbol{v})=(\boldsymbol{v}^{n}\cdot\boldsymbol{\nabla})\boldsymbol{v}^{n+1}.$$

Before continuing with the finite element discretization the weak form has to be obtained by projecting the above equations onto a space of weighting function $\boldsymbol{w}\in\mathcal{V}$ for the momentum equation and $q\in\mathcal{Q}$ for the incompressibility condition. To do so, neglecting the boundary terms for simplification purposes, find the velocity $\boldsymbol{v}^{n+1}\in\mathcal{S}$ and the pressure $p^{n+1}\in\mathcal{Q}$, such that for all $(\boldsymbol{w},q)\in\mathcal{V}\times\mathcal{Q}$ and where both functional spaces $\mathcal{S}$ and $\mathcal{V}$ verify the prescribed boundary conditions $\boldsymbol{v}^{n+1}=\boldsymbol{v}_D^{n+1}$ on $\partial\Omega$:

$$\begin{aligned}&\left(\boldsymbol{w},\frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^{n}}{\Delta t}\right)+c(\boldsymbol{v}^{n};\boldsymbol{w},\boldsymbol{v}^{n+1})+a(\boldsymbol{w},\boldsymbol{v}^{n+1})+b(\boldsymbol{w},p^{n+1})=(\boldsymbol{w},\boldsymbol{f}^{n+1})\\&b(\boldsymbol{v}^{n+1},q)=0.\end{aligned}$$

Then, the finite element discretization can be written as follows:

$$\begin{aligned}&\mathbf{M}\left(\frac{\mathbf{v}^{n+1}-\mathbf{v}^{n}}{\Delta t}\right)+(\mathbf{C}(\mathbf{v}^{n})+\mathbf{K})\mathbf{v}^{n+1}+\mathbf{G}^{T}\mathbf{p}^{n+1}=\mathbf{f}^{n+1}\\&\mathbf{G}\mathbf{v}^{n+1}=\mathbf{0}.\end{aligned}$$

Discretization of the Chorin-Temam projection method

Now the Chorin-Temam projection scheme is described as an example for a fractional-step method. The idea of fractional step approaches is to divide the numerically complex time integration into simpler substeps. Generally, a time-dependent problem of the form

$$\frac{\partial w}{\partial t}+\mathcal{L}w=f\quad\text{where}\quad\mathcal{L}w=\mathcal{L}_1w+\mathcal{L}_2w$$

can be divided into two steps. In case of the Chorin-Temam projection method the first step looks as follows:

$$\begin{aligned}&\frac{\boldsymbol{v}_{\text{int}}^{n+1}-\boldsymbol{v}^{n}}{\Delta t}+(\boldsymbol{v}^{*}\cdot\boldsymbol{\nabla})\boldsymbol{v}^{**}-\nu\nabla^{2}\boldsymbol{v}^{**}=\boldsymbol{f}^{n+1}&&\text{in }\Omega\\&\boldsymbol{v}_{\text{int}}^{n+1}=\boldsymbol{v}_D^{n+1}&&\text{on }\partial\Omega.\end{aligned}$$

To further reduce the numerical costs the convective term is again replaced by a semi-explicit approximation:

$$\boldsymbol{v}^* = \boldsymbol{v}^n \quad \text{and} \quad \boldsymbol{v}^{**} = \boldsymbol{v}_{\text{int}}^{n+1}.$$

Before proceeding with the finite element discretization the weak form of the first step equations has to be constructed. Find the intermediate velocity $\boldsymbol{v}_{\text{int}}^{n+1} \in \mathcal{S}_{\text{int}}$, such that for all $\boldsymbol{w} \in \mathcal{V}_{\text{int}}$ and where both functional spaces $\mathcal{S}_{\text{int}}$ and $\mathcal{V}_{\text{int}}$ satisfy the Dirichlet boundary conditions $\boldsymbol{v}_{\text{int}}^{n+1} = \boldsymbol{v}_D^{n+1}$ on $\partial\Omega$:

$$\left(\boldsymbol{w}, \frac{\boldsymbol{v}_{\text{int}}^{n+1} - \boldsymbol{v}^n}{\Delta t}\right) + c(\boldsymbol{v}^*; \boldsymbol{w}, \boldsymbol{v}^{**}) + a(\boldsymbol{w}, \boldsymbol{v}^{**}) = (\boldsymbol{w}, \boldsymbol{f}^{n+1}).$$

From this weak form the finite element discretization can be derived:

$$\mathbf{M}_1 \left(\frac{\mathbf{v}_{\text{int}}^{n+1} - \mathbf{v}^n}{\Delta t}\right) + (\mathbf{C}(\mathbf{v}^*) + \mathbf{K})\mathbf{v}_{\text{int}}^{n+1} = \mathbf{f}^{n+1}.$$

The second step contains the pressure term and the incompressibility equation of the Navier-Stokes problem:

$$\begin{aligned}
&\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}_{\text{int}}^{n+1}}{\Delta t} + \boldsymbol{\nabla} p^{n+1} = 0 && \text{in } \Omega \\
&\boldsymbol{\nabla} \cdot \boldsymbol{v}^{n+1} = 0 && \text{in } \Omega \\
&\boldsymbol{n} \cdot \boldsymbol{v}^{n+1} = \boldsymbol{n} \cdot \boldsymbol{v}_D^{n+1} && \text{on } \partial\Omega.
\end{aligned}$$

Due to the Helmholtz decomposition principle $\boldsymbol{v} = \boldsymbol{v_s} + \boldsymbol{\nabla}\phi$, where $\boldsymbol{v_s}$ is a solenoidal field such that $\boldsymbol{n} \cdot \boldsymbol{v_s} = 0$, only the normal component of the velocity can be prescribed on the boundary. This becomes clearer when rewriting the first equation of the second step as

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}_{\text{int}}^{n+1} - \Delta t \boldsymbol{\nabla} p^{n+1} \quad \text{or} \quad \boldsymbol{v}_{\text{int}}^{n+1} = \boldsymbol{v}^{n+1} + \Delta t \boldsymbol{\nabla} p^{n+1},$$

where $\boldsymbol{v}_{\text{int}}^{n+1}$ can be understood as a composition of the divergence free velocity field $\boldsymbol{v}^{n+1}$ and the gradient of a scalar function $-\Delta t \boldsymbol{\nabla} p^{n+1}$. Even though only applying the normal component of the velocity on the boundary leads to a spurious pressure boundary layer, it does not prevent proper convergence of the algorithm as can be seen in the results later. In general the second step can be see as the projection of the intermediate velocity onto the solenoidal space:

$$\boldsymbol{v}^{n+1} = \mathbb{P}\boldsymbol{v}_{\text{int}}^{n+1} \quad \text{and} \quad \Delta t \boldsymbol{\nabla} p^{n+1} = (\mathbf{I} - \mathbb{P})\boldsymbol{v}_{\text{int}}^{n+1}.$$

The weak form is given by finding the end-of-step velocity $\boldsymbol{v}^{n+1} \in \mathcal{S}$ and the pressure $p^{n+1} \in \mathcal{Q}$, such that for all $(\boldsymbol{w}, q) \in \mathcal{V} \times \mathcal{Q}$ and where both functional spaces $\mathcal{S}$ and $\mathcal{V}$ verify the prescribed boundary conditions $\boldsymbol{n} \cdot \boldsymbol{v}^{n+1} = \boldsymbol{n} \cdot \boldsymbol{v}_D^{n+1}$ on $\partial\Omega$:

$$\left(\boldsymbol{w}, \frac{\boldsymbol{v}_{\text{int}}^{n+1} - \boldsymbol{v}^n}{\Delta t}\right) + b(\boldsymbol{w}, p^{n+1}) = 0$$
$$b(\boldsymbol{v}^{n+1}, q) = 0.$$

Deriving the finite element discretization from the weak form yields:

$$\mathbf{M}_2 \left(\frac{\mathbf{v}_{\text{int}}^{n+1} - \mathbf{v}^n}{\Delta t}\right) + \mathbf{G}^T \mathbf{p}^{n+1} = \mathbf{0}$$
$$\mathbf{G}\mathbf{v}^{n+1} = \mathbf{0},$$

or in matrix form:

$$\begin{pmatrix} \mathbf{M}_2/\Delta t & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}^{n+1} \\ \mathbf{p}^{p+1} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_2 \mathbf{v}_{\text{int}}^{n+1}/\Delta t \\ \mathbf{h} \end{pmatrix}.$$

This system resulting from the second step can be solved analogous to the Stoke problem by applying two steps. First, computing the pressure field

$$(\mathbf{G}^T \mathbf{M}_2^{-1} \mathbf{G}) \mathbf{p}^{n+1} = \frac{1}{\Delta t} \mathbf{G}^T \mathbf{v}_{\text{int}}^{n+1},$$

and finally computing the end-of-step velocity

$$\mathbf{M}_2 \mathbf{v}^{n+1} = \mathbf{M}_2 v_{\text{int}}^{n+1} - \Delta t \mathbf{G} \mathbf{p}^{n+1}.$$

Results

In the following section the results obtained by the presented methods are discussed. The following constants were used during the execution of the computations:

- Reynold's number: $Re = 100$
- Kinematic viscosity: $\nu = 1/Re$
- Time step size: $\Delta t = 0.01$
- Number of time steps: $n_t = 100$

Furthermore, a uniform mesh of the size 10x10 with Q2Q1-elements was applied.

Figure 1 shows the pressure field for both method at time $t = 1.0$. Both show a smooth pressure surface over the whole domain with the expected peaks at both upper corners of the domain.

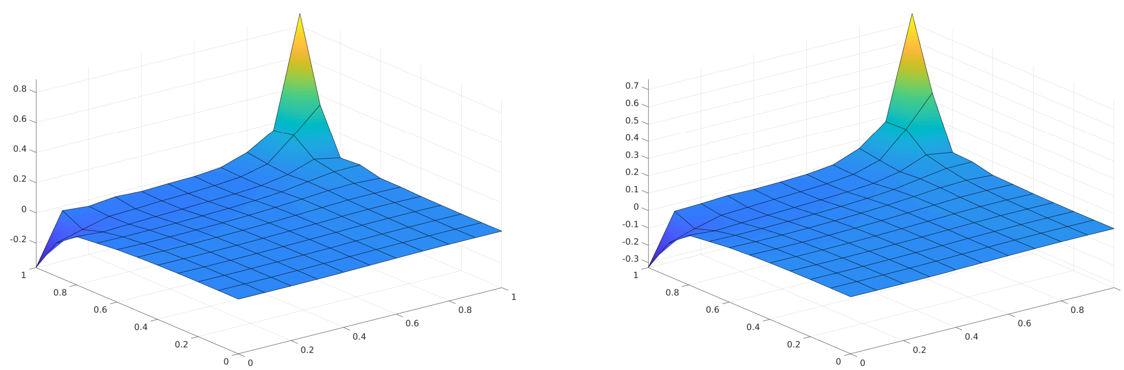


Figure 1: Pressure field of the semi-implicit first order monolithic scheme (left) and the Chorin-Temam projection method (right) at time $t = 1.00$

When looking at the comparison of the velocity fields at time $t = 1.0$ in Figure 2 a slight difference is notable. The size of the velocity vector for the Chorin-Temam projection are slightly larger implying a higher velocity of the fluid in the upper right corner of the domain.

Figures 3 and 4 show the development of the streamlines over time. We can observe that they develop in a similar way, but that in case of the semi-implicit first order monolithic scheme the streamlines extend further into the lower part of the domain. Also the density of the streamlines at the boarders of the domain is higher for the Chorin-Temam projection method.

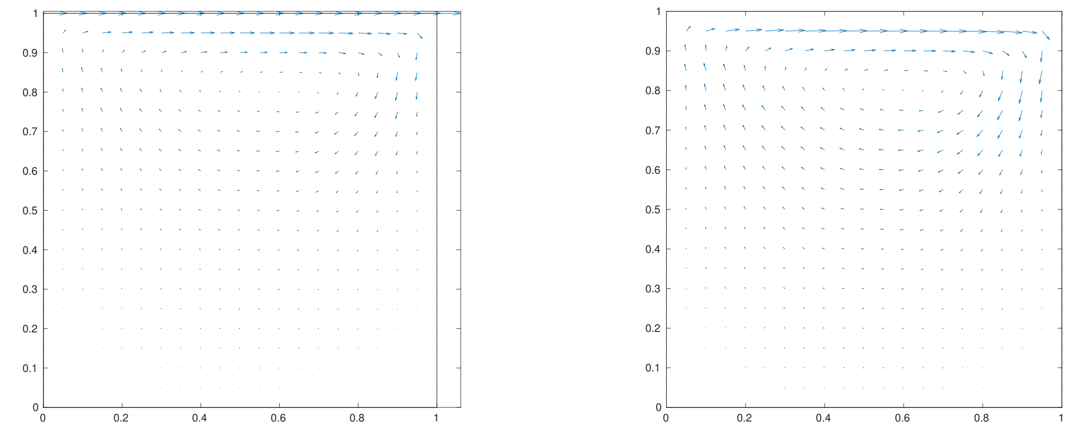
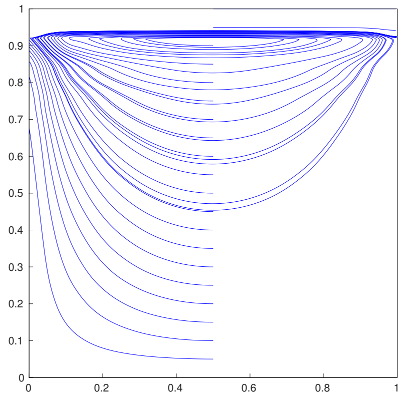
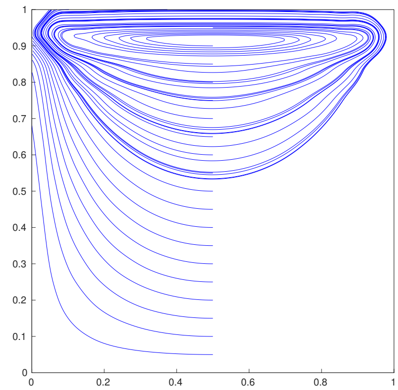


Figure 2: Velocity field of the semi-implicit first order monolithic scheme (left) and the Chorin-Temam projection method (right) at time $t = 1.00$

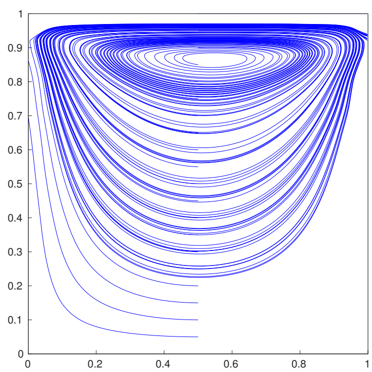
Additionally, the computational time for both methods was measured. The results showed that the semi-implicit first order monolithic scheme was about 25% faster than the Chorin-Temam projection method. This is probably caused by the additional computation of second step, which was described in the previous section. Though, from literature it is known that the lag in speed is compensated with a higher accuracy. With this in mind the differences in the presented results can be explained.



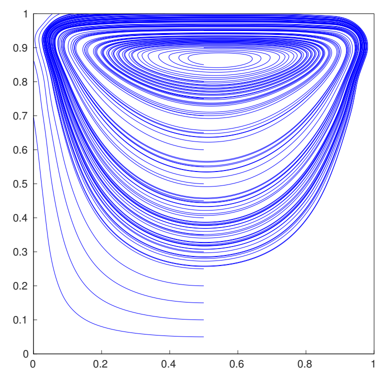
(a) Semi-implicit first order monolithic scheme, $t = 0.01$



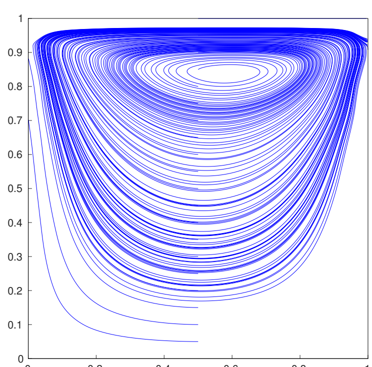
(b) Chorin-Temam projection method, $t = 0.01$



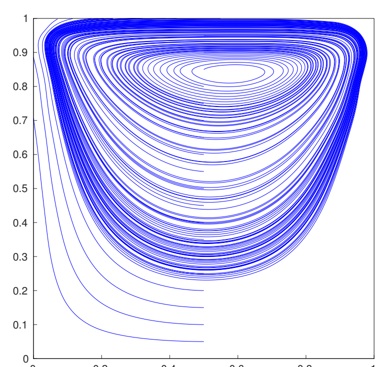
(c) Semi-implicit first order monolithic scheme, $t = 0.25$



(d) Chorin-Temam projection method, $t = 0.25$

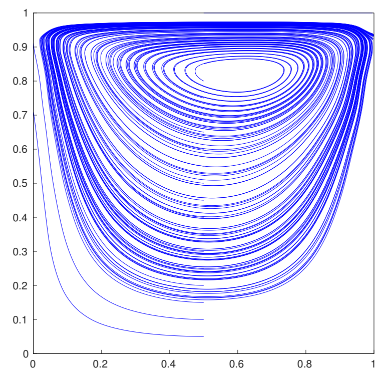


(e) Semi-implicit first order monolithic scheme, $t = 0.50$

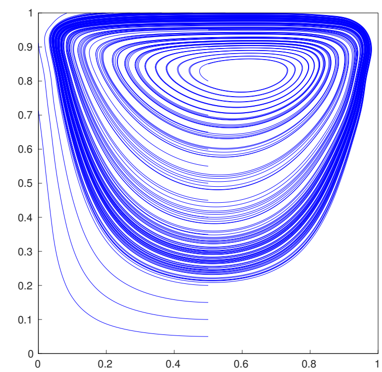


(f) Chorin-Temam projection method, $t = 0.50$

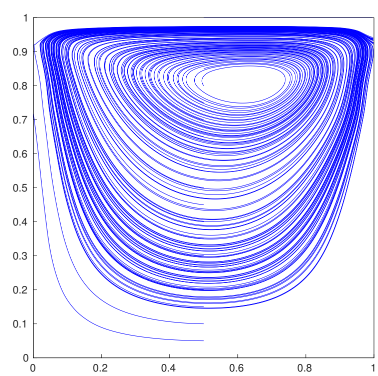
Figure 3: Streamlines of the semi-implicit first order monolithic scheme (left) and the Chorin-Temam projection method (right) at different points of time



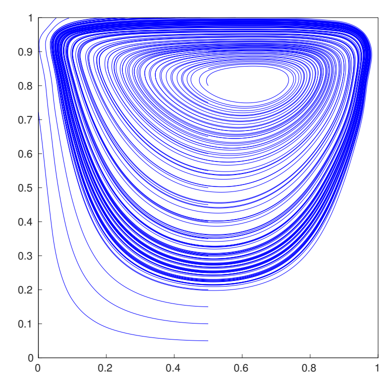
(a) Semi-implicit first order monolithic scheme, $t = 0.75$



(b) Chorin-Temam projection method, $t = 0.75$



(c) Semi-implicit first order monolithic scheme, $t = 1.00$



(d) Chorin-Temam projection method, $t = 1.00$

Figure 4: Streamlines of the semi-implicit first order monolithic scheme (left) and the Chorin-Temam projection method (right) at different points of time