Linear Elements

a. Galerkin

$$\int w(a.\nabla u)\, d\Omega + \int \nabla w.(\nu \nabla u) d\Omega + \int w\sigma u\, d\Omega = \int ws\, d\Omega$$

So, we only need to solve

Convection matrix C = $\int N_a(a.\nabla N_b) d\Omega$

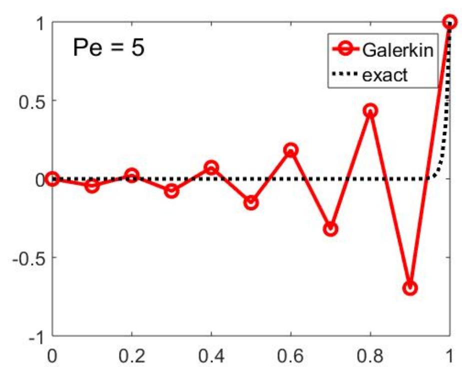
Reaction Matrix R = $\int N_a \sigma N_b d\Omega \approx \mathbf{0}$: for linear element this term will be neglected

Diffusion Matrix D = $\int \nabla N_a(\nu \nabla N_b) d\Omega$

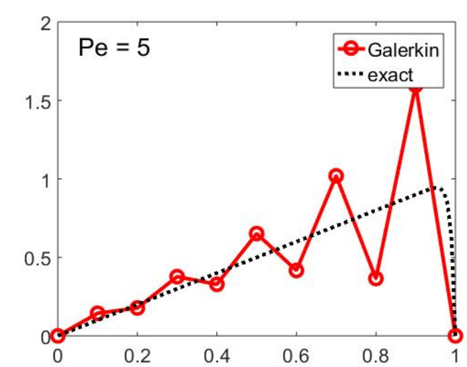
Source vector SV = $\int N_a s\, d\Omega$

Response:-

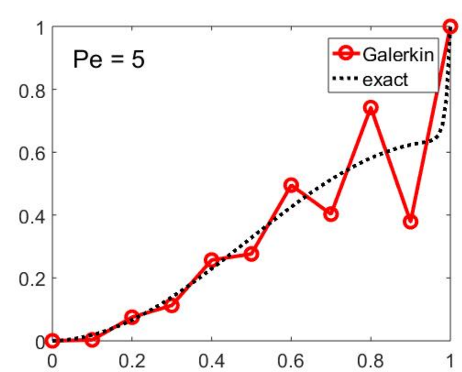
(i) Problem-1
s=0, a=1, ν=0.01



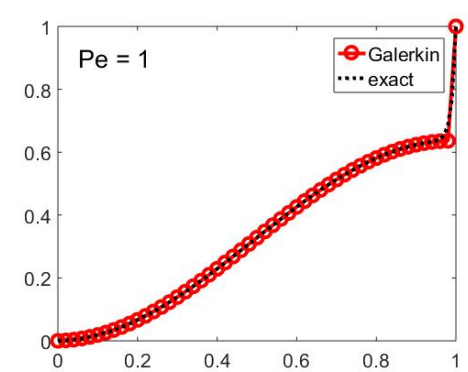
(ii) Problem-2
s=1, a=1, ν=0.01



(iii) Problem-3
s= sin (πx), a=1, ν=0.01

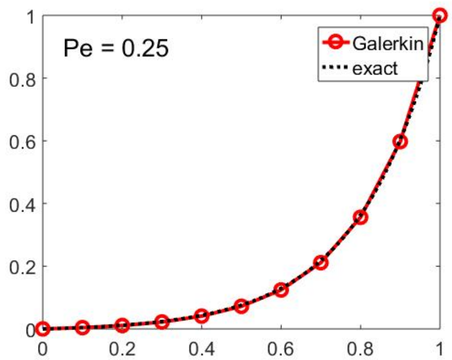


(iv) Problem-3
s= sin (πx), a=1, ν=0.01



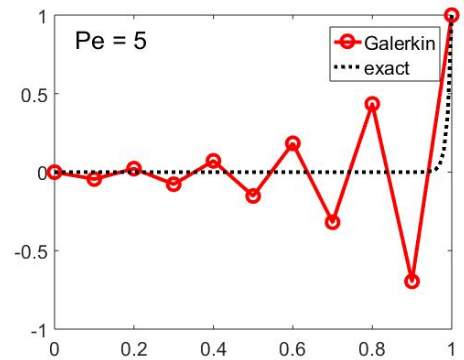
(v) Problem-1

s=0, a=1, ν=0.2,



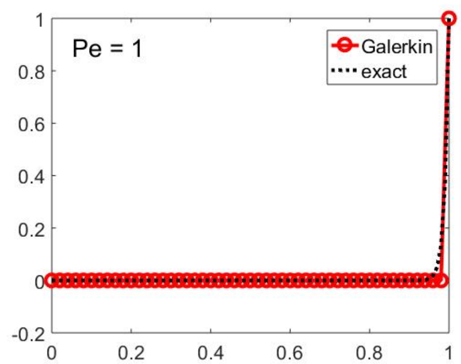
(vi) Problem-1

s=0, a=20, ν=0.2



(vii) Problem-1

s=1, a=1, ν=0.01



<u>Comments</u>:

Case (i),(ii,(iii) & (vi),as Pe number is equal to 5 (Pe ≥ 1) that's why Galerkin exploded and solution oscillate at nodes. To overcome oscillation we should have a finer mesh and reduce the element size such that Pe number approaches Pe ≤.1. As it is evident from case (iv) & (vii) where Pe=1, the Galerkin produces approximate solution near to exact solution. In Case (v), Galerkin produces almost exact solution with sufficient amount of diffusion added to the problem.

b. Upwind Stream (SU)

$$\int \mathrm{w}(\mathrm{a}.\nabla u)\,\mathrm{d}\Omega + \int \nabla \mathrm{w}.(\nu\nabla u)\mathrm{d}\Omega + \int \mathrm{w}\sigma u\,\mathrm{d}\Omega + \int \acute{\upsilon}/\mathrm{IaI}^{\wedge}2(\mathrm{a}.\nabla \mathrm{w})(\mathrm{a}.\nabla u)\,\mathrm{d}\Omega \;=\; \int \mathrm{ws}\,\mathrm{d}\Omega$$

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Reaction Matrix R = $\int N_a \sigma N_b \mathrm{d}\Omega \approx \mathbf{0}$: for linear element this term will be neglected

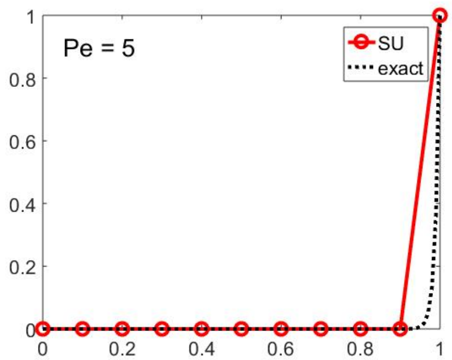
Diffusion Matrix D = $\int \nabla N_a(\nu\nabla N_b)\mathrm{d}\Omega$

Added Artificial Diffusion Matrix D' = $\int \boldsymbol{\tau}(\mathrm{a}.\nabla N_a)(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

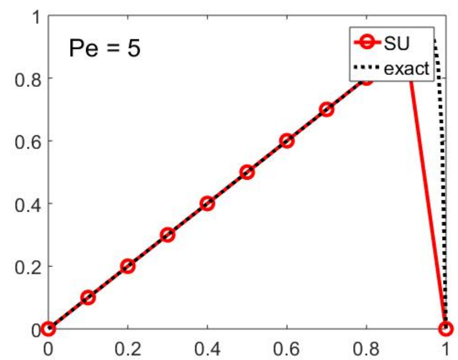
Source vector SV = $\int N_a \mathrm{s}\, \mathrm{d}\Omega$

<u>Response:</u>

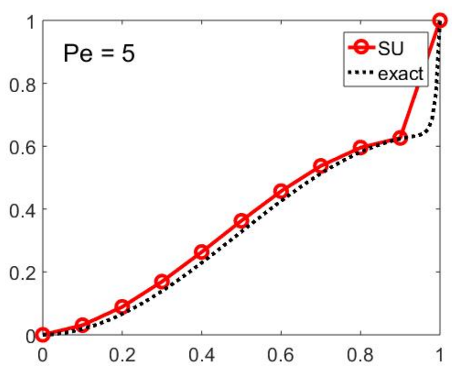
(i) Problem-1
s=0, a=1, ν=0.01



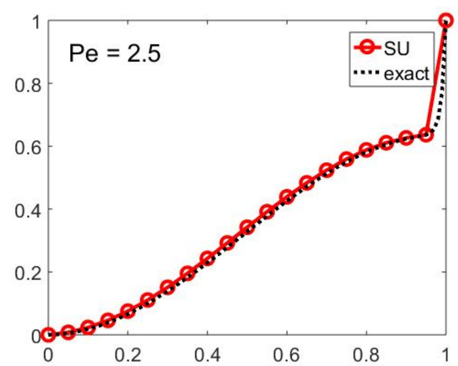
(ii) Problem-2
s=1, a=1, ν=0.01



(iii) Problem-3
s= sin (πx), a=1, ν=0.01



(iv) Problem-3
s= sin (πx), a=1, ν=0.01



<u>Comments</u>:
Case (i),(ii) & (iii) artificial diffusion is added to problem to balance the dominated convection. It smoothen the solution and it's not consistent and diffusion is only added in the upwind direction and it pushes the solution to converge in that direction as illustrated by above results. We can still produce nearly exact solution by reducing the Pe number (finer mesh) Case-(iv).

c. Upwind Stream Petrov-Galerkin(SUPG)

$$\int \mathrm{w}(\mathrm{a}.\nabla \mathcal{U})\, \mathrm{d}\Omega + \int \nabla \mathrm{w}.(\nu \nabla \mathcal{U})\mathrm{d}\Omega + \int \mathrm{w}\sigma\mathcal{U}\, \mathrm{d}\Omega + \sum \big(\boldsymbol{\mathcal{P}}(\boldsymbol{w})\boldsymbol{\tau}, \boldsymbol{\mathcal{R}}(\mathbf{u})\big) = \int \mathrm{ws}\, \mathrm{d}\Omega + \sum (\boldsymbol{\mathcal{P}}(\boldsymbol{w})\boldsymbol{\tau s})$$

Adding stabilization terms, now we have to solve.

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Reaction Matrix R = $\int N_a \sigma N_b \mathrm{d}\Omega \approx \mathbf{0}$ for linear element this term will be neglected

Diffusion Matrix D = $\int \nabla N_a(\nu \nabla N_b)\mathrm{d}\Omega$

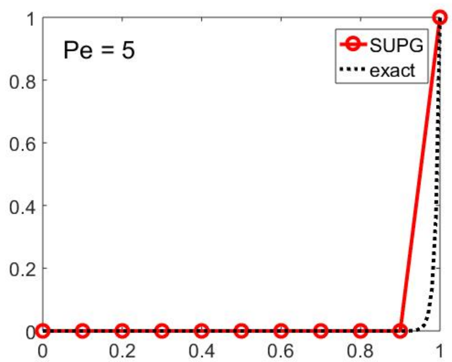
Stabilization Matrix S = $\sum(\mathcal{P}(w)\tau\mathcal{R}(\mathrm{u}) \approx \sum(\mathrm{a}.\nabla N_a)\tau(\mathrm{a}.\nabla N_b)$ for linear element

Source vector SV = $\int N_a \mathrm{s}\, \mathrm{d}\Omega + \sum(\mathcal{P}(w)\tau \mathrm{s}) \approx \int N_a \mathrm{s}\, \mathrm{d}\Omega + \sum((\mathrm{a}.\nabla N_a)\tau \mathrm{s})$ for linear element

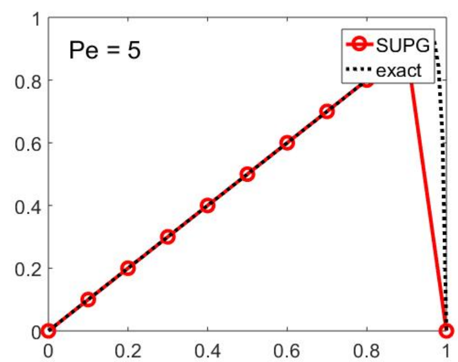
Where $\mathcal{P}(w) = (\mathrm{a}.\nabla N_a)$ and $\mathcal{R}(\mathrm{u}) = (\mathrm{a}.\nabla N_b) - \nabla.(\nu \nabla N_b) + \sigma \mathrm{u}$

<u>Response:-</u>

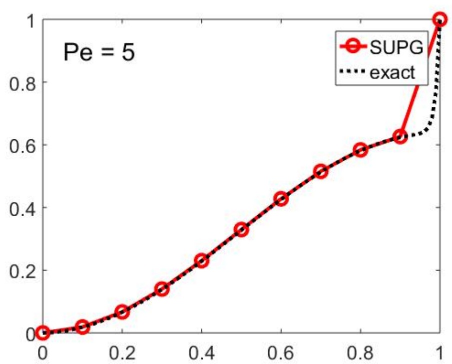
(i) Problem-1
s=0, a=1, ν=0.01



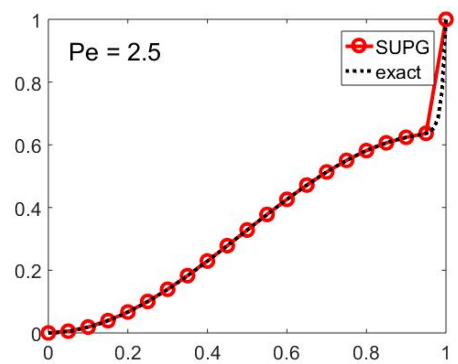
(ii) Problem-2
s=1, a=1, ν=0.01



(iii) Problem-3
s= sin (πx), a=1, ν=0.01



(iv) Problem-3
s= sin (πx), a=1, ν=0.01



<u>Comments:</u>

Stabilization is added to ensure the consistency and we have a more accurate approximate solution with course mesh.

d. Galerkin Least Square (GLS)

$$\int \mathrm{w}(\mathrm{a}.\nabla\mathcal{U})\,\mathrm{d}\Omega + \int \nabla\mathrm{w}.(\nu\nabla\mathcal{U})\mathrm{d}\Omega + \int \mathrm{w}\sigma\mathcal{U}\,\mathrm{d}\Omega + \sum\big(\mathcal{L}(\boldsymbol{w})\boldsymbol{\tau},\boldsymbol{\mathcal{R}}(\mathbf{u})\big) = \int \mathrm{ws}\,\mathrm{d}\Omega + \sum(\mathcal{L}(\boldsymbol{w})\boldsymbol{\tau s})$$

Adding stabilization terms, now we have to solve.

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Reaction Matrix R = $\int N_a\sigma N_b \mathrm{d}\Omega \approx \mathbf{0}$ for linear element this term will be neglected

Diffusion Matrix D = $\int \nabla N_a(\nu\nabla N_b)\mathrm{d}\Omega$

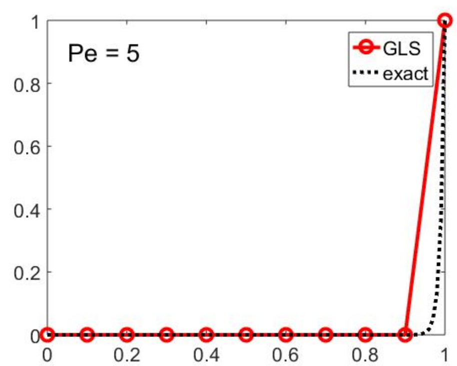
Stabilization Matrix S = $\sum(\mathcal{L}(w)\tau\mathcal{R}(\mathrm{u}) \approx \sum(\mathrm{a}.\nabla N_a)\tau(\mathrm{a}.\nabla N_b)$ for linear element

Source vector SV = $\int N_a \mathrm{s}\,\mathrm{d}\Omega + \sum(\mathcal{L}(w)\tau\mathrm{s}) \approx \int N_a \mathrm{s}\,\mathrm{d}\Omega + \sum((\mathrm{a}.\nabla N_a)\tau\mathrm{s})$ for linear element

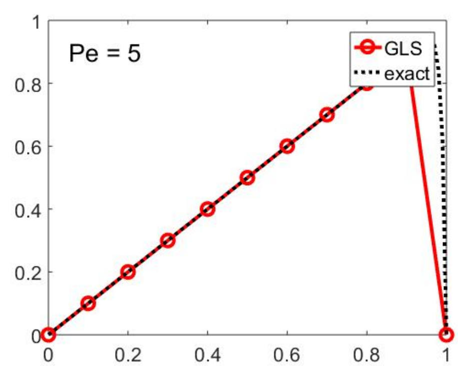
Where $\mathcal{L}(\mathrm{w}) = (\mathrm{a}.\nabla N_a) - \nabla.(\nu\nabla N_a) + \sigma\mathrm{w}$ and $\mathcal{R}(\mathrm{u}) = (\mathrm{a}.\nabla N_b) - \nabla.(\nu\nabla N_b) + \sigma\mathrm{u}$

Response:-

(i) Problem-1
s=0, a=1, ν=0.01

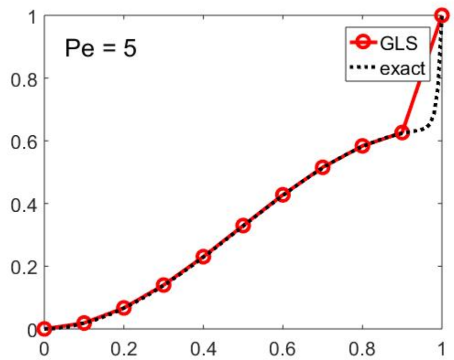


(ii) Problem-2
s=1, a=1, ν=0.01



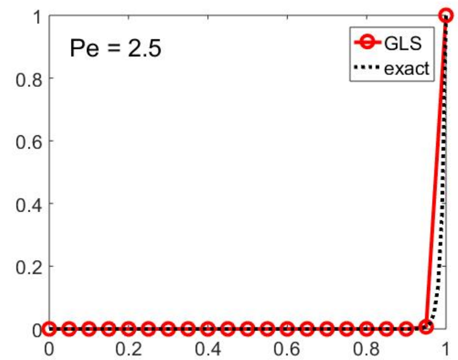
(iii) Problem-3

s= sin (πx), a=1, ν=0.01



(iv) Problem-1

s= 0, a=1, ν=0.01



<u>Comments:</u>

For linear elements, SUPG and GLS produce same approximate solutions.

Quadratic Elements

a. Galerkin

$$\int \mathrm{w}(\mathrm{a}.\nabla \mathcal{U})\, \mathrm{d}\Omega + \int \nabla \mathrm{w}.(\nu \nabla \mathcal{U}) \mathrm{d}\Omega + \int \mathrm{w}\sigma \mathcal{U}\, \mathrm{d}\Omega = \int \mathrm{ws}\, \mathrm{d}\Omega$$

So, we only need to solve

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

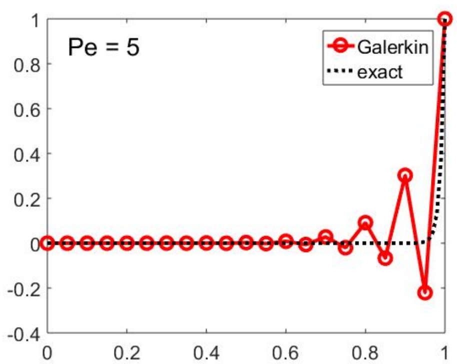
Reaction Matrix R = $\int N_a \sigma N_b \mathrm{d}\Omega$

Diffusion Matrix D = $\int \nabla N_a(\nu \nabla N_b)\mathrm{d}\Omega$

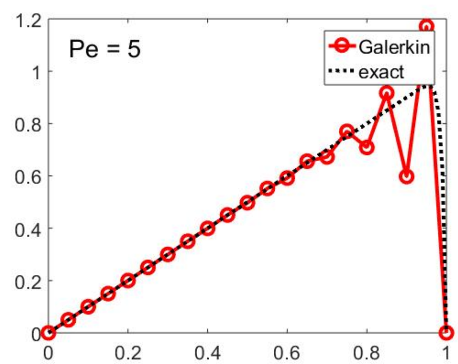
Source vector SV = $\int N_a s\, \mathrm{d}\Omega$

Response:-

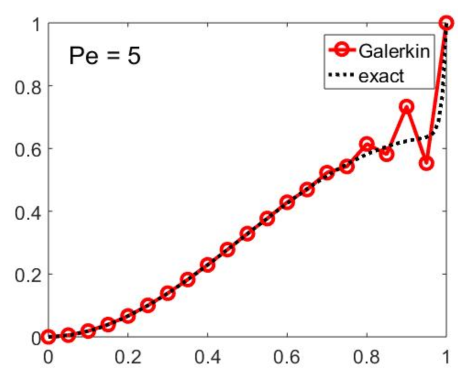
(i) Problem-1
s=0, a=1, ν=0.01



(ii) Problem-2
s=1, a=1, ν=0.01



(iii) Problem-3
s= sin (πx), a=1, ν=0.01



Comments:

Quadratic elements shared a third node in middle in a sense to decrease the node to node distance and play a role to reduce the oscillations at nodes while having the same Pe number. So in comparison to linear elements, Galerkin produces a relatively stable approximate solution with same no of elements and oscillations appear in the convective dominated region only.

b. Upwind Stream (SU)

$$\int \mathrm{w}(\mathrm{a}.\nabla u)\,\mathrm{d}\Omega + \int \nabla\mathrm{w}.(\nu\nabla u)\mathrm{d}\Omega + \int \mathrm{w}\sigma u\,\mathrm{d}\Omega + \int \acute{\upsilon}/\mathrm{IaI}\hat{}2(\mathrm{a}.\nabla\mathrm{w})(\mathrm{a}.\nabla u)\,\mathrm{d}\Omega = \int \mathrm{ws}\,\mathrm{d}\Omega$$

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Reaction Matrix R = $\int N_a \sigma N_b \mathrm{d}\Omega$

Diffusion Matrix D = $\int \nabla N_a(\nu\nabla N_b)\mathrm{d}\Omega$

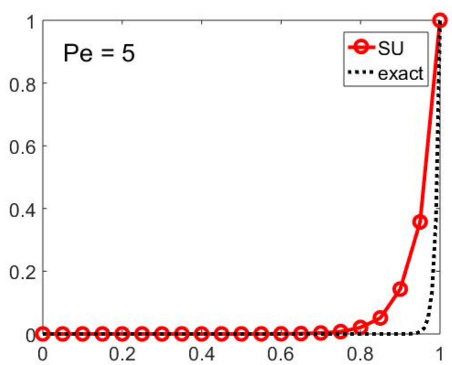
Added Artificial Diffusion Matrix D' = $\int \boldsymbol{\tau}(\mathrm{a}.\nabla N_a)(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Source vector SV = $\int N_a \mathrm{s}\,\mathrm{d}\Omega$

<u>Response:</u>

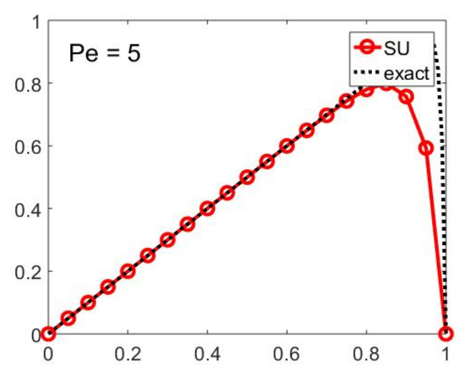
(i) Problem-1

s=0, a=1, ν=0.01



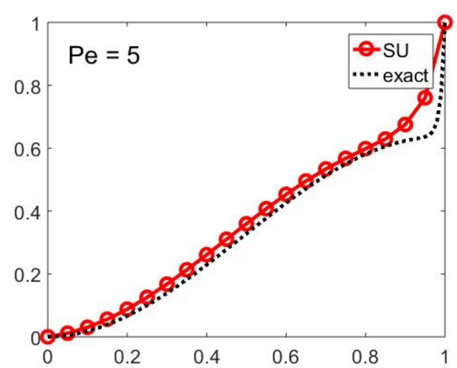
(ii) Problem-2

s=1, a=1, ν=0.01



(iii) Problem-3

s= sin (πx), a=1, ν=0.01



Comments:
Quadrilateral elements produce better results with added diffusion in wind direction than the linear elements but still approximate solution is not consistent.

c. Upwind Stream Petrov-Galerkin(SUPG)

$$\int w(a.\nabla\mathcal{U})\,d\Omega + \int \nabla w.(\nu\nabla\mathcal{U})d\Omega + \int w\sigma\mathcal{U}\,d\Omega + \sum\left(\boldsymbol{\mathcal{P}}(\boldsymbol{w})\boldsymbol{\tau},\boldsymbol{\mathcal{R}}(\mathbf{u})\right) = \int ws\,d\Omega + \sum(\boldsymbol{\mathcal{P}}(\boldsymbol{w})\boldsymbol{\tau s})$$

Adding stabilization terms, now we have to solve.

Convection matrix C = $\int N_a(a.\nabla N_b)d\Omega$

Reaction Matrix R = $\int N_a\sigma N_b d\Omega$

Diffusion Matrix D = $\int \nabla N_a(\nu\nabla N_b)d\Omega$

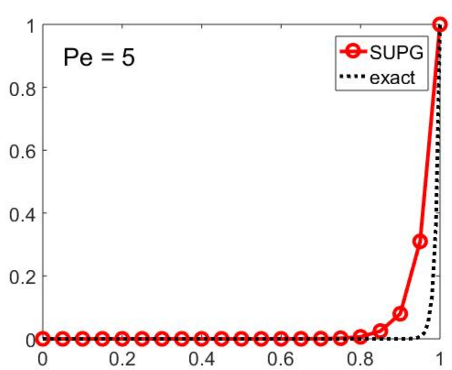
Stabilization Matrix S = $\sum(\mathcal{P}(w)\tau\mathcal{R}(u)$

Source vector SV = $\int N_a s\,d\Omega + \sum(\mathcal{P}(w)\tau s)$

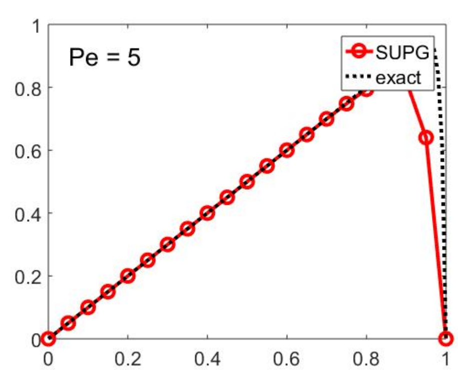
Where $\mathcal{P}(w) = (a.\nabla N_a)$ and $\mathcal{R}(u) = (a.\nabla N_b) - \nabla.(\nu\nabla N_b) + \sigma u$

Response:-

(i) Problem-1
s=0, a=1, ν=0.01

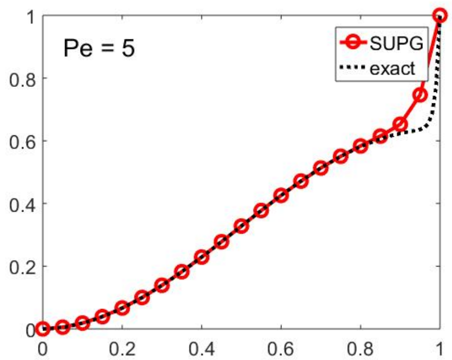


(ii) Problem-2
s=1, a=1, ν=0.01



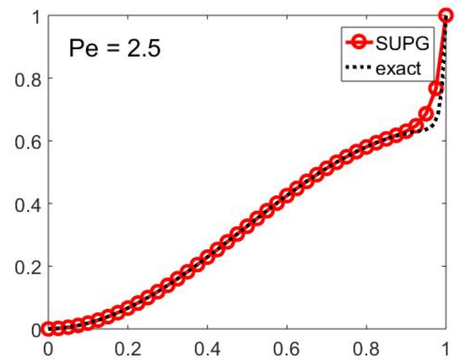
(iii) Problem-3

s= sin (πx), a=1, ν=0.01



(iv) Problem-1

s= sin (πx), a=1, ν=0.01



Comments:

Stabilization with quad elements are more accurate than the linear elements and still more better results can be achieved by finer mesh Case-(iv).

d. Galerkin Least Square (GLS)

$$\int \mathrm{w}(\mathrm{a}.\nabla \mathcal{U})\,\mathrm{d}\Omega + \int \nabla \mathrm{w}.(\nu \nabla \mathcal{U})\mathrm{d}\Omega + \int \mathrm{w}\sigma\mathcal{U}\,\mathrm{d}\Omega + \sum \big(\mathcal{L}(\boldsymbol{w})\boldsymbol{\tau}, \boldsymbol{\mathcal{R}}(\mathbf{u})\big) = \int \mathrm{ws}\,\mathrm{d}\Omega + \sum (\mathcal{L}(\boldsymbol{w})\boldsymbol{\tau s})$$

Adding stabilization terms, now we have to solve.

Convection matrix C = $\int N_a(\mathrm{a}.\nabla N_b)\mathrm{d}\Omega$

Reaction Matrix R = $\int N_a \sigma N_b \mathrm{d}\Omega$

Diffusion Matrix D = $\int \nabla N_a(\nu \nabla N_b)\mathrm{d}\Omega$

Stabilization Matrix S = $\sum(\mathcal{L}(w)\tau\mathcal{R}(\mathrm{u})$

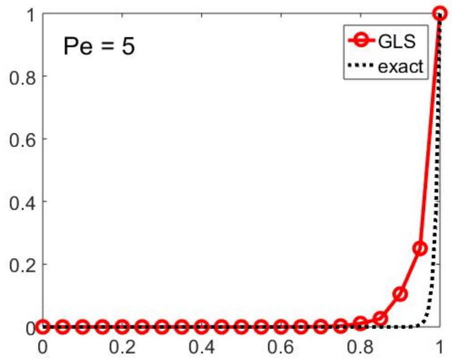
Source vector SV = $\int N_a \mathrm{s}\,\mathrm{d}\Omega + \sum(\mathcal{L}(w)\tau \mathrm{s})$

Where $\mathcal{L}(\mathrm{w}) = (\mathrm{a}.\nabla N_a) - \nabla.(\nu\nabla N_a) + \sigma\mathrm{w}$ and $\mathcal{R}(\mathrm{u}) = (\mathrm{a}.\nabla N_b) - \nabla.(\nu\nabla N_b) + \sigma\mathrm{u}$

Response:-

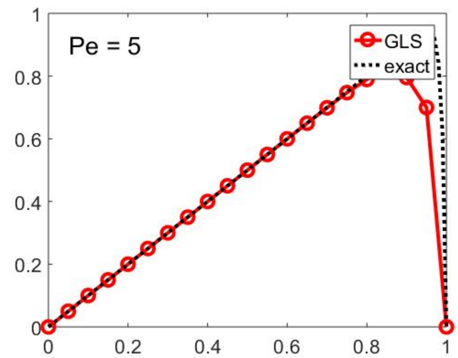
Problem-1

s=0, a=1, ν=0.01



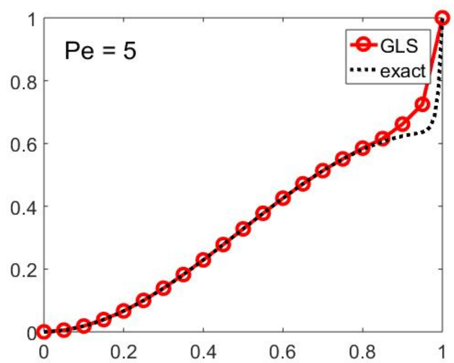
Problem-2

s=1, a=1, ν=0.01



Problem-3

s=sin (πx), a=1, ν=0.01



Comments:

With Quad elements, both SUPG and GLS behave exactly the same and that is strange.

ADNAN ALI IMRAN

HOME WORK #2 FEF ①

Discretization (Linear Elements)

Strong Form

$$a \cdot \nabla u - \nabla \cdot (\nu \nabla u) - \sigma u = s \quad \Omega$$

$$u(0) = 0 \; , \quad u(1) = 1$$

Weak Form by Galerkin Approach

$$\int_\Omega w (a \cdot \nabla u) \, d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) \, d\Omega + \int_\Omega w \sigma u \, d\Omega = \int_\Omega w s \, d\Omega$$

Adding Artificial Diffusion

$$\int_\Omega w (a \cdot \nabla u) d\Omega + \int_\Omega \nabla w \cdot ((\nu + \bar{\nu}) \nabla u) \, d\Omega + \int_\Omega w \sigma u \, d\Omega = \int_\Omega w s \, d\Omega$$

Streamline Upwind SU

$$\int_\Omega w (a \cdot \nabla u) \, d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) \, d\Omega + \int_\Omega w \sigma u \, d\Omega$$

$$+ \int_\Omega \frac{\bar{\nu}}{\|a\|^2} (a \cdot \nabla w)(a \cdot \nabla u) \, d\Omega = \int_\Omega w s \, d\Omega$$

Stabilized Consistent Formulation

$$\int_\Omega w (a \cdot \nabla u) \, d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) \, d\Omega + \int_\Omega w \sigma u \, d\Omega$$

$$+ \sum_e \int_{\Omega_e} P(w) \, \tau \, R(u) \, d\Omega = \int_\Omega w s \, d\Omega$$

Streamline upwind Petrov-Galerkin (SUPG)

$$P(w) = a.\nabla w$$

$$\int_{\Omega} w\,(a.\nabla u)\,d\Omega + \int_{\Omega} \nabla w.(\nu \nabla u)\,d\Omega + \int_{\Omega} w\,\sigma u\,d\Omega$$

$$+ \sum_e \int_{\Omega_e} (a.\nabla w)\,\tau\,\big((a.\nabla u) - \nabla.(\nu\nabla u) + \sigma u\big)\,d\Omega$$

$$= \int_{\Omega} ws\,d\Omega + \sum_e \int_{\Omega_e} (a.\nabla w)\,\tau s\,d\Omega$$

Galerkin Least-Squares (GLS)

$$P(w) = L(w) = a.\nabla w - \nabla.(\nu\nabla w) + \sigma w$$

$$\int_{\Omega} w(a.\nabla u)\,d\Omega + \int_{\Omega} \nabla w.(\nu\nabla u)\,d\Omega + \int_{\Omega} w\sigma u\,d\Omega$$

$$+ \sum_e \int_{\Omega_e} (a.\nabla w - \nabla.(\nu\nabla w) + \sigma w)\,\tau\,(a.\nabla u - \nabla.(\nu\nabla u) + \sigma u)\,d\Omega$$

$$= \int_{\Omega} ws\,d\Omega + \sum_e \int_{\Omega_e} (a.\nabla w - \nabla.(\nu\nabla w) + \sigma w)\,\tau s\,d\Omega$$

Discretization of Linear 1D element

For Galerkin Approach $\quad \because \sigma u = 0$ for linear Element.

$$\int_0^l \sum_{B=1}^{mEl+1} \left(a N_A \frac{\partial N_B}{\partial x} + \nu \frac{\partial N_A}{\partial x}\frac{\partial N_B}{\partial x} \right) dx = \int_0^l N_A s\,dx$$

As 1D element

each element with 2-nodes.

Connectivity Matrix =

1	2
2	3
3	4
⋮	⋮

∵ Code

```
mPt = mElem + 1;
X = (dom(1):h:dom(2));
T = [1:mPt-1; 2:mPt]';
```

Shape functions

$$N_1(\xi) = \frac{1}{2}(1-\xi) \quad , \quad N_2(\xi) = \frac{1}{2}(1+\xi) \qquad \because -1 \leq \xi \leq 1$$

So, $u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$

and $x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2$

For uniform mesh size 'h' $\Rightarrow$ $dx = \frac{1}{2}(x_2 - x_1)\,d\xi = \frac{h}{2}\,d\xi$

then $$\frac{\partial N_b}{\partial x} = \frac{\partial N_b}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{2}{h}\frac{\partial N_b}{\partial \xi} \qquad \text{for } b = 1,2$$

So, C^e matrix will be

$$C^e = a\int_{\Omega^e} \begin{pmatrix} N_1 \frac{\partial N_1}{\partial x} & N_1 \frac{\partial N_2}{\partial x} \\ N_2 \frac{\partial N_1}{\partial x} & N_2 \frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{a}{2}\begin{pmatrix} -1 & +1 \\ -1 & +1 \end{pmatrix}$$

and

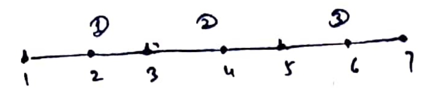
$$K^e = v\int_{\Omega^e} \begin{pmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{v}{h}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$$

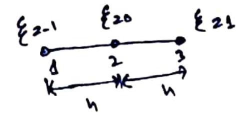
and $$f^e = \int_{\Omega^e} \left\{ N_1(N_1 s_1 + N_2 s_2) \,,\; N_2(N_1 s_1 + N_2 s_2) \right\}^T dx$$

As $(C^e + K^e)\,u = f^e$ for element.

Discretization of Quadratic 1D Elements for Galerkin Approach

$\because \xi = -1 \leq \xi \leq 1$





Connectivity Matrix T =

1	2	3
3	4	5
5	6	7
⋮	⋮	⋮

∵ Code ⇒

```
nPt = nElem * p+1;
X = (dom(1) : h/2 : dom(2));
T = [1:2:nPt-2; 2:2:nPt-1; 3:2:nPt]';
```

Shape functions ⇒ $N_1(\xi) = \frac{1}{2}\xi(\xi-1)$, $N_2(\xi) = 1-\xi^2$, $N_3(\xi) = \frac{1}{2}\xi(\xi+1)$

So, $u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3$ and

$$x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3$$

with uniform mesh $dx = h\,d\xi$ and

$$\frac{\partial N_b}{\partial x} = \frac{\partial N_b}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{h}\frac{\partial N_b}{\partial \xi}, \quad b = 1,2,3$$

So,

$$C^e = a\int_{\Omega^e}\begin{pmatrix} N_1\frac{\partial N_1}{\partial x} & N_1\frac{\partial N_2}{\partial x} & N_1\frac{\partial N_3}{\partial x} \\ N_2\frac{\partial N_1}{\partial x} & N_2\frac{\partial N_2}{\partial x} & N_2\frac{\partial N_3}{\partial x} \\ N_3\cdot\frac{\partial N_1}{\partial x} & N_3\frac{\partial N_2}{\partial x} & N_3\frac{\partial N_3}{\partial x}\end{pmatrix} dx$$

$$C^e = \frac{a}{2}\begin{pmatrix} -1 & 4/3 & -1/3 \\ -4/3 & 0 & 4/3 \\ 1/3 & -4/3 & 1 \end{pmatrix}$$

and K^e

$$K^e = \nu \int_{\Omega^e} \begin{pmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x} \end{pmatrix} dx$$

$$K^e = \frac{\nu}{6h} \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

and f^e

$$f^e = \int_{\Omega} N_b (N_1 S_1 + N_2 S_2 + N_3 S_3)\, dx \qquad b = 1,2,3$$

$$(C^e + K^e)\, u = f^e \quad \text{for element.}$$