

MatLab Exercise 1

Steady Advection-Diffusion Equation

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1 Introduction

This report will cover the 1D *steady state advection-diffusion* equation, seen in (1), solution via *FEM* with emphasis on stabilization methods proposed as exercises in laboratory session 1. The first part will briefly explain the implementation of each stabilization method: Streamline Upwind Petrov Galerkin (SUPG) and Galerkin Least Square (GLS) for both 1D linear and quadratic elements. Following, results will be discussed.

$$au_x - \nu u_{xx} = s \tag{1}$$

1.1 Implementation for quadratic elements

In order to enable the use of quadratic elements in the formulation of the problem some small changes were made in the code provided. Firstly, in the MatLab file named as *main.m* the definition of the connectivity matrix for a mesh with 3-noded elements was done T , as shown.

```
if p==1  
T = [1:nPt-1; 2:nPt]';  
elseif p ==2  
T = [1:nPt-2; 2:nPt-1; 3:nPt]';
```

This enabled the use of quadratic elements ($p = 2$ in the code). As quadratic elements features shaping functions with second derivatives different than zero, that will appear in stabilization terms, the MatLab files named as *SUPG-system.m* and *GLS-system.m* had the following lines added to the code.

```
N2xi = referenceElement.N2xi; % In variables declaration part  
N2x_ig = N2xi(ig,:)*2/h; % Inside de loop of Gauss-Points (stiffness matrix)
```

This enabled the computation of those second order derivatives.

1.2 Implementation SUPG and GLS stabilization methods

For the implementation of those methods, as seen in bibliography, some extra terms needs to be added in the elemental stiffness matrices and vector forces. This implementation was done in such a way to enable the application of those methods for 1D linear and quadratic elements. This way, terms with second order derivative were *not* neglected in each formulations. As a result, for the SUPG method the resultant line for the evaluation of the elemental stiffness matrix in the *SUPG_system.m* code was as follow. Were the bold terms correspond to the added terms.

$$Ke = Ke + w_{ig}*(N_{ig}'*a*Nx_{ig} + Nx_{ig}'*nu*Nx_{ig}) ... \\ + w_{ig}*(\tau*a*Nx_{ig})'*(a*Nx_{ig})-w_{ig}*(\tau*a*Nx_{ig}')*(nu*N2x_{ig});$$

Te force term became:

$$fe = fe + w_{ig}*(N_{ig})'*s+w_{ig}*(\tau*a*Nx_{ig}')*s;$$

For the GLS method the resultant coding line for evaluation of the stiffness matrix became:

$$Ke = Ke + w_{ig}*(N_{ig}'*a*Nx_{ig} + Nx_{ig}'*nu*Nx_{ig}) ... \\ + w_{ig}*(\tau*a*Nx_{ig})'*(a*Nx_{ig})-w_{ig}*(\tau*a*Nx_{ig}')*(nu*N2x_{ig})... \\ -w_{ig}*(\tau*nu*N2x_{ig}')*(a*Nx_{ig})+w_{ig}*(\tau*nu*N2x_{ig}')*(nu*N2x_{ig});$$

Te force term became:

$$fe = fe + w_{ig}*(N_{ig})'*s+w_{ig}*(\tau*a*Nx_{ig}')*s-w_{ig}*(\tau*nu*N2x_{ig}')*s;$$

2 Results

2.1 Exercise I: Galerkin Method

The first example ($u(0) = 0$, $u(1) = 1$ and $s = 0$), were run using Galerkin Method (already provided) and four different set of values for convective coefficient (a), diffusion coefficient (ν) and number of elements (N).Figure 1 shows the results.

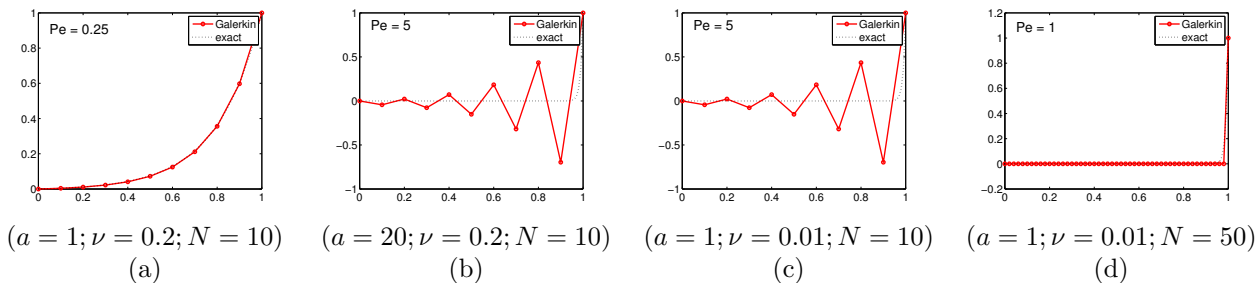


Figure 1: Galerkin Method.

As expected the Galerkin method presented oscillations for Peclet values higher than one ($Pe > 1$), which is the reason for the use of stabilizing methods.

2.2 Exercise II: Comparison Galerkin, SU,SUPG and GLS

The Galerkin, SU, SUPG and GLS methods are compared using example 3 ($u(0) = 0$, $u(1) = 1$ and $s = \sin x$) for values of $a = 1$, $\nu = 0.01$ and $N = 10$.In Figure 2 the results are presented.

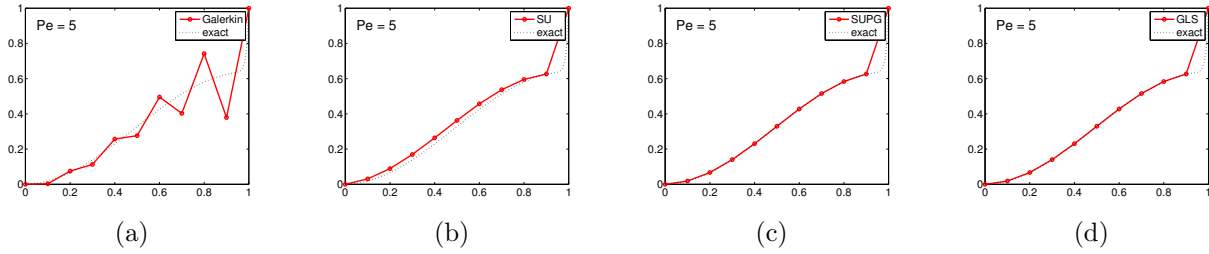


Figure 2: Comparison: example 3.

As it can be seen, again Galerkin method presented high oscillations due to Pe number, or its lack of enough diffusion. The SU method deviates considerably from the exact solution. This behavior is noticeable when the source term is not constant as its formulation is non-consistent one in terms application of the numerical diffusion, which is employed only in the strong form equation (different problem is being solved). The SUPG and GLS had the almost same behavior in this scenario, as both are consistent and ensure that the exact solution is also solution of the weak formulation.

2.3 Exercise III: Quadratic elements

The Galerkin, SU, SUPG and GLS methods are compared using example 3 ($u(0) = 0, u(1) = 1$ and $s = \sin x$) for values of $a = 1, \nu = 0.01$ and $N = 10$, this time with quadratic elements. In Figure 2 the results are presented.

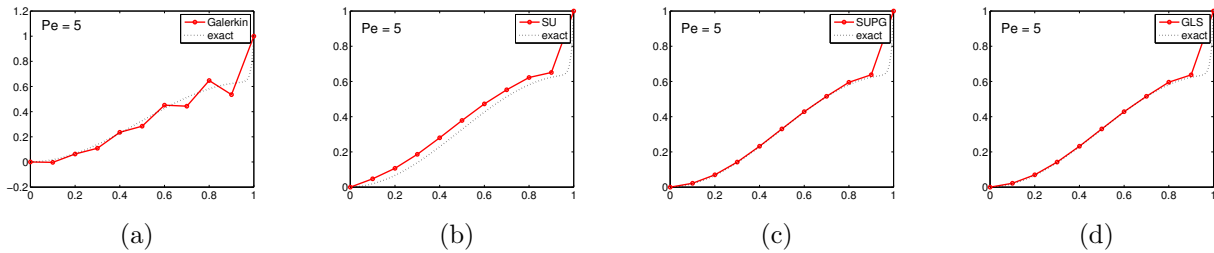


Figure 3: Comparison: example 3 (quadratic elements).

For a quadratic formulation, Galerkin method is improved (less oscillations), while SU has a worst behavior when compared with the linear elements formulation. It can be observed that for SUPG, Figure 3 (c), and GLS, Figure 3 (d), the solution is not exact at nodes anymore. This is due to the fact that the "optimal" diffusion value for exact solution at nodes was derived based on the 1D linear formulation, and not quadratic one.

2.4 Exercise IV: Variation of τ

The effect of variation of τ was tested for the method SUPG using again example 3 ($u(0) = 0, u(1) = 1$ and $s = \sin x$) for values of $a = 1, \nu = 0.01$ and $N = 10$. In Figure 3 the results are presented.

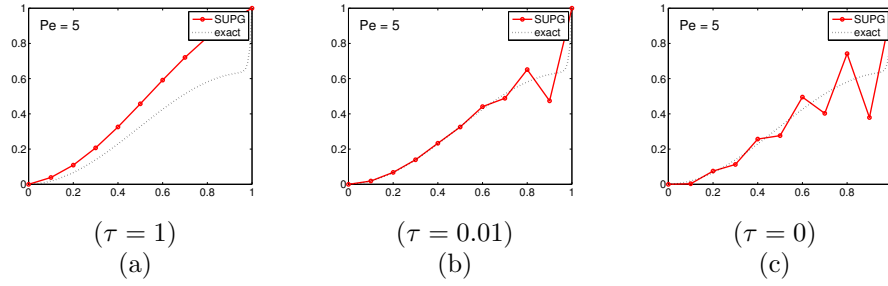


Figure 4: Variation of τ (SUPG)

The SUPG method behaves as expected. When $\tau = 1$ which is higher than the "optimal" value, too much diffusion is added. As it can be seen in Figure 4 (a), the solution is smoothed-out against the advection direction. For $\tau = 0.01$, the diffusion is too small (lower than optimal) and oscillations appear. In the limit when $\tau \rightarrow 0$ the SUPG solution is the same as Galerkin one (Figure 2 (a))