Eugenio José Muttio Zavala — TEF

Homework - Assignment 1

Obtain Non-Conservative Form of N-S Equations

- $\frac{\partial(\rho\bar{v})}{\partial t} + \nabla\cdot(\rho v\otimes v - \sigma) = \rho b$ — Conservative Form of N-S equations

$$\underbrace{\rho\frac{\partial\bar{v}}{\partial t}}_{1} + \underbrace{\bar{v}\frac{\partial\rho}{\partial t}}_{2} + \underbrace{\nabla\cdot(\rho v\otimes v)}_{3} - \underbrace{\nabla\cdot\sigma}_{4} = \underbrace{\rho b}_{5} \qquad (1.1)$$

- Now considering the next identity in order to modify 3

$$\nabla\cdot(\rho\bar{v}\otimes\bar{v}) = \left((\nabla\bar{v})\bar{v} + \bar{v}\cdot\nabla\bar{v}\right)\rho + \bar{v}(\bar{v}\cdot\nabla\rho)$$

Using an identity

$$\nabla(\rho\bar{v}) = \rho\nabla\bar{v} + \bar{v}\cdot\nabla\rho$$

$$\bar{v}\cdot\nabla\rho = \nabla\cdot(\rho\bar{v}) - \rho\nabla\bar{v}$$

$$\nabla\cdot(\rho\bar{v}\otimes\bar{v}) = \rho\left(\cancel{(\nabla\bar{v})\bar{v}} + \bar{v}\cdot\nabla\bar{v}\right) + \bar{v}\left[\nabla\cdot(\rho\bar{v}) - \cancel{\rho\nabla\bar{v}}\right]$$

$$= \rho\bar{v}\cdot\nabla\bar{v} + \bar{v}\,\nabla\cdot(\rho\bar{v})$$

- Substituting in (1.1)

$$\rho\frac{\partial\bar{v}}{\partial t} + \bar{v}\frac{\partial\rho}{\partial t} + \rho\bar{v}\cdot\nabla\bar{v} + \bar{v}\,\nabla\cdot(\rho\bar{v}) - \nabla\cdot\sigma = \rho b$$

- Rearranging terms

$$\underbrace{\bar{v}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{v})\right]}_{\text{Continuity Equation} = 0} + \rho\frac{\partial\bar{v}}{\partial t} + \rho\bar{v}\cdot\nabla\bar{v} - \nabla\cdot\sigma = \rho b$$

$$\rho\frac{\partial\bar{v}}{\partial t} + \rho(\bar{v}\cdot\nabla)\bar{v} - \nabla\cdot\sigma = \rho b \qquad (2)$$

Non Conservative Form of N-S Equation (General)

• Now, considering the next constitutive equation:

$$\boxed{\sigma = -pI + 2\mu \nabla^s \bar{v}}$$

• For Newtonian Incompressible fluids.

Substituting this expression in ② and dividing by ρ

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho (\bar{v} \cdot \nabla)\bar{v} - \nabla \cdot (-pI + 2\mu \nabla^s \bar{v}) = \rho b$$

$\hookrightarrow \boxed{\nu = \frac{\mu}{\rho}}$

$$\downarrow$$

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} \underbrace{- 2\nu \nabla \cdot \nabla^s \bar{v}} + \nabla p = b$$

$$\nabla^s \bar{v} = \frac{1}{2}(\nabla \bar{v} + \nabla^T \bar{v})$$

$$\therefore \ 2\nu \nabla \cdot \nabla^s \bar{v} = \frac{2}{2}\nu\left(\nabla^2 \bar{v} + \nabla(\nabla \cdot \bar{v})\right)$$

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} - \nu \nabla^2 \bar{v} + \underbrace{\nabla(\nabla \cdot \bar{v})} + \nabla p = b$$

If incompressibility holds

$$\cancel{\frac{\partial \rho}{\partial t}}^{0} + \nabla \cdot \bar{v} = 0 \rightarrow \underline{\underline{\nabla \cdot \bar{v} = 0}}$$

$\therefore$

$$\boxed{\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} - \nu \nabla^2 \bar{v} + \nabla p = b}$$

Non Conservative Form of Navier-Stokes Equation for Newtonian Incompressible Fluids