Finite Elements in Fluid

Homework 2: Steady-transport examples

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1. INTRODUCTION

The convection-diffusion-reaction equations is

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \boldsymbol{a} \bullet \nabla u - \nabla \bullet (\nu \nabla u) + \sigma u = s$$

where u:unknown; a:convection/transport velocity; ν :diffusivity; σ :reaction coefficient s:source term

The numerical treatment of convection

$$\boldsymbol{a} \bullet \nabla u - \nabla \bullet (\nu \nabla u) = s \ \ in\ \Omega$$
$$u = u_D \ on\ \Gamma_D$$
$$\nu \frac{\partial \mathrm{u}}{\partial \mathrm{n}} = u_N$$

Galerkin weak form

$$\int_\Omega w(\boldsymbol{a} \bullet \nabla u) d\Omega - \int_\Omega w \nabla \bullet (\nu \nabla u)\, d\Omega = \int_\Omega ws\, d\Omega \quad \text{for all } \mathrm{w} \in \nu$$

Take the shape function N_i, obtain

$$u(x) \cong u(x)^h = \sum_{i=1}^{n} N_i\,(x) u_i$$

Than get the numerical solution.

2. OBJECTIVE

2.1 1D convection-diffusion equation with constant coefficients and Dirichlet boundary condition:

$$au_x - \nu u_{xx} = s \ x \in [0,1]$$

$$u(0) = u_0; u(1) = u_1$$

3 examples: 1) $\mathrm{s} = 0,\ u_0 = 0,\ u_1 = 1$

2) $\mathrm{s} = 1,\ u_0 = 0,\ u_1 = 0$

3) $\mathrm{s} = \sin(\pi \mathrm{x}),\ u_0 = 0,\ u_1 = 1$

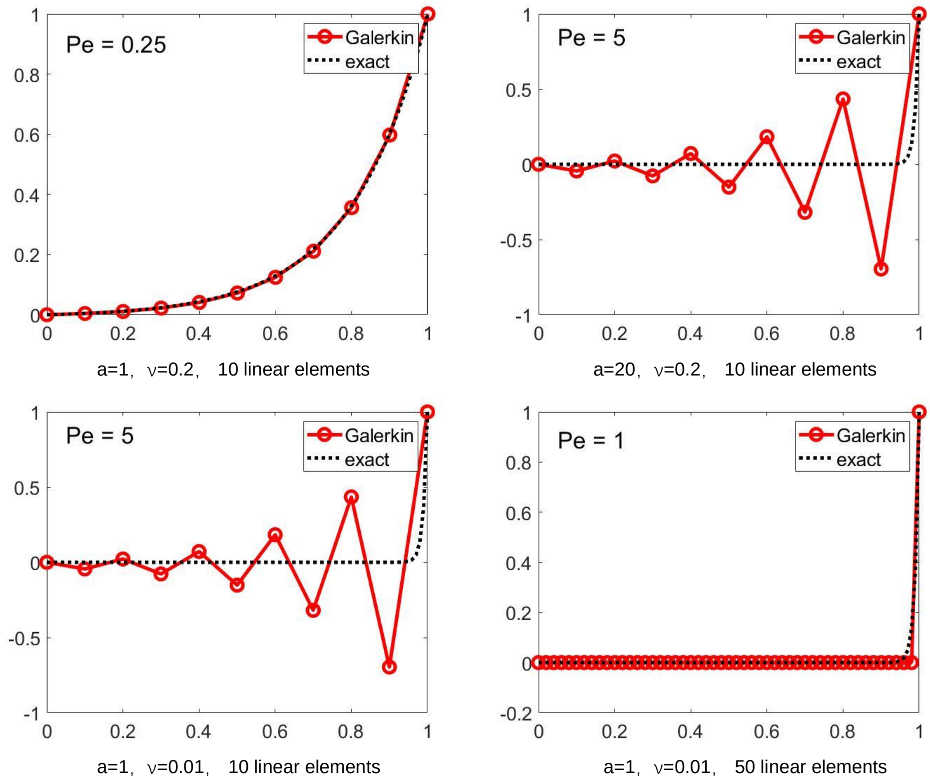
2.2 Quadratic elements

2.3 Source term

3. METHODOLOGY AND RESULTS

3.1 Linear elements, solve the first example using Galerkin's method with

$$\mathrm{a} = 1, \nu = 0.2, 10\ linear\ elements$$
$$\mathrm{a} = 20, \nu = 0.2, 10\ linear\ elements$$
$$\mathrm{a} = 1, \nu = 0.01, 10\ linear\ elements$$
$$\mathrm{a} = 1, \nu = 0.01, 50\ linear\ elements$$



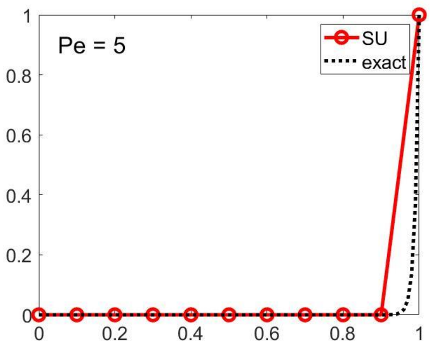
From the 4 cases above, we can find that the Galerkin method is not suitable for each case. While Peclet number $\mathrm{Pe} = \frac{\mathrm{ah}}{2\nu} > 1$, Galerkin method lacks enough diffusion and the numerical solution shows oscillation. That means Galerkin method lacks diffusion when convection dominates. So, we can consider give more weight to terms associated with transport in the upwind direction.

This problem's exact solution is

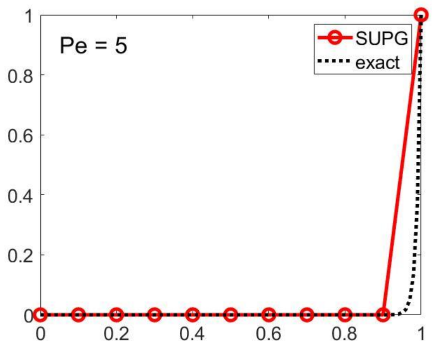
$$\mathrm{y(x)} = \frac{1 - e^{\frac{ax}{\nu}}}{1 - e^{\frac{a}{\nu}}}$$

3.2 Solve the third case using, Streamline upwind, SUPG, GLS with the optimal stabilization parameter.

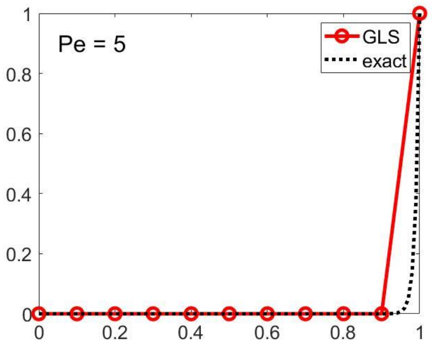
Exmaple1, $s = 0, u_0 = 0, u_1 = 1,$



Stream upwind

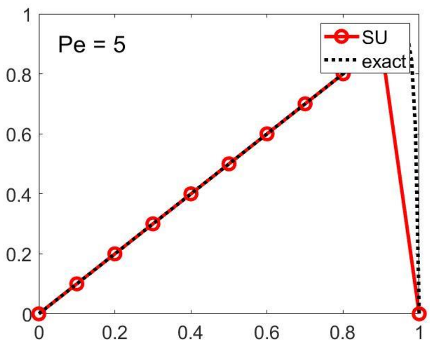


Stream upwind Petrov-Galerkin

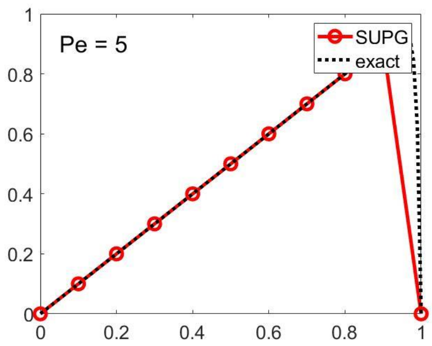


Galerkin/ Least-Squares

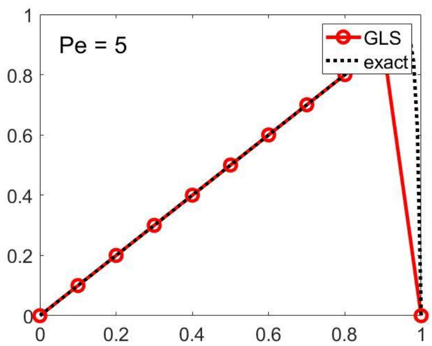
Exmaple2, $s = 1, u_0 = 0, u_1 = 0,$



Stream upwind

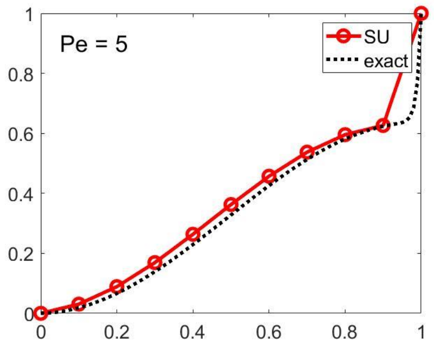


Stream upwind Petrov-Galerkin

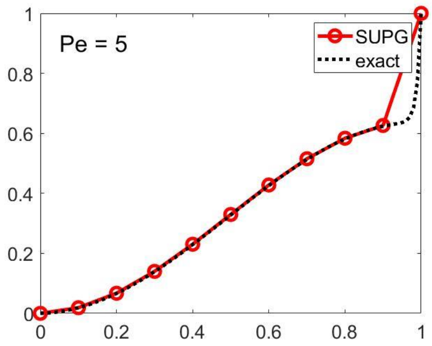


Galerkin/ Least-Squares

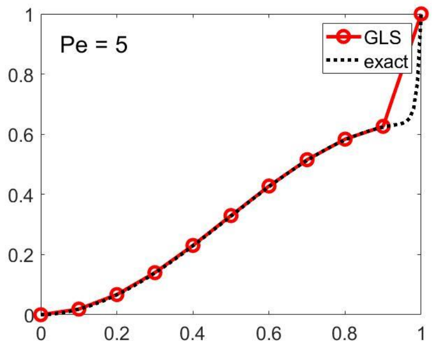
Exmaple3, $s = \sin(\pi x)$, $u_0 = 0$, $u_1 = 1$



Stream upwind

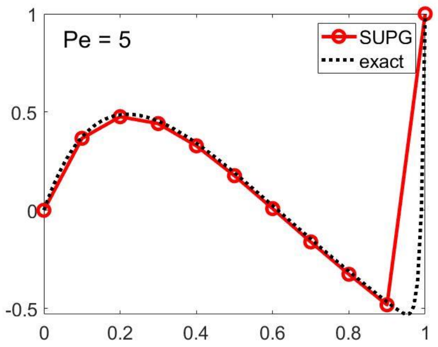


Stream upwind Petrov-Galerkin

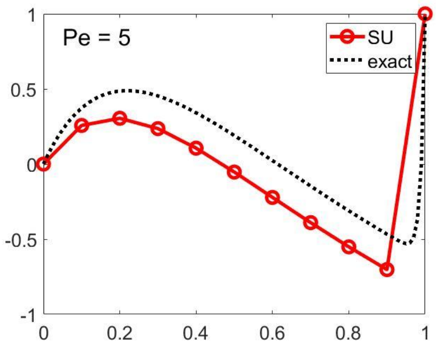


Galerkin/ Least-Squares

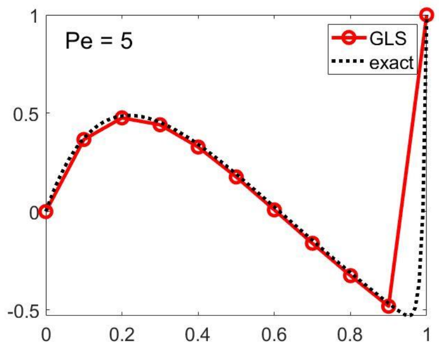
Exmaple4, $s = 10e^{-5x} - 4e^{-x}$, $u_0 = 0$, $u_1 = 1$



Stream upwind



Stream upwind Petrov-Galerkin



Galerkin/ Least-Squares

The SU method have smooth solution for high Peclet number. If there is constant coefficient and there is no variable source term, the solution is good. But this method can not fit for example 3 and example 4.

Compared with SU, the SUPG performs better while source term is variable term. For stabilization parameter $\tau = \bar{\nu}/\|a\|^2$, the weighting function coincides with SU however in SUPG the perturbation is applied consistently to all terms.

While the added stabilization term is not symmetric, it is mean technical difficulties to ensure stability. GLS squares stabilization technique overcomes this drawback adding a symmetric stabilization term. For the linear elements, the SUPG and the GLS actually is the same one, since the second-order derivative terms are zero. $\nabla\bullet(\nu \nabla w) = 0$

3.3 Quadratic elements

We could give a choice to select element order as input.

```
% Reference element: numerical quadrature and shape functions
p = cinput('Choose: 1: linear elements, 2: quadratic elements= ', 1);
referenceElement = SetRefereceElement(p);
```

And then we should prepare the quadratic element.

```
if p == 1
nPt = nElem + 1;
h = (dom(2) - dom(1))/nElem;
X = (dom(1):h:dom(2))';
T = [1:nPt-1; 2:nPt]';
elseif p == 2
nPt = 2*nElem +1;
h = (dom(2) - dom(1))/(2*nElem);
X = (dom(1):h:dom(2))';
T = [1:2:2*nElem-1; 2:2:2*nElem; 3:2:2*nElem+1]';
end
```

The second order derivative of shape function should be code as
N2x_ig=N2xi(ig,:)*2/(h^2);

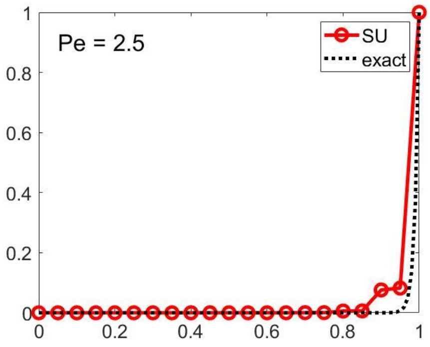
For this way, the stiffness matrix K in SUPG method should be code as

```
Ke=Ke+w_ig*(N_ig'*(a*Nx_ig)+Nx_ig'*(nu*Nx_ig))
        +(tau*a*Nx_ig)'*(a*Nx_ig-nu*N2x_ig));
```

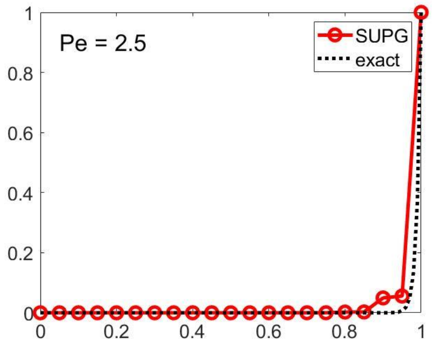
For GLS,

```
Ke=Ke+w_ig*(N_ig'*(a*Nx_ig)+Nx_ig'*(nu*Nx_ig))
        +(a*Nx_ig'-N2x_ig')*tau*(a*Nx_ig-nu*N2x_ig));
```

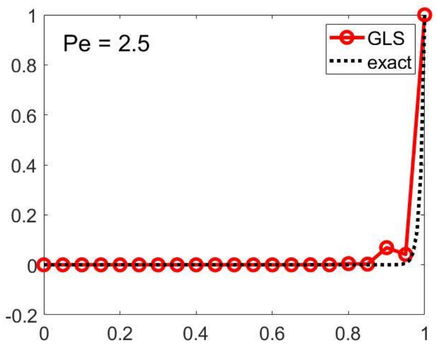
Exmaple1, $s = 0, u_0 = 0, u_1 = 1,$



Stream upwind

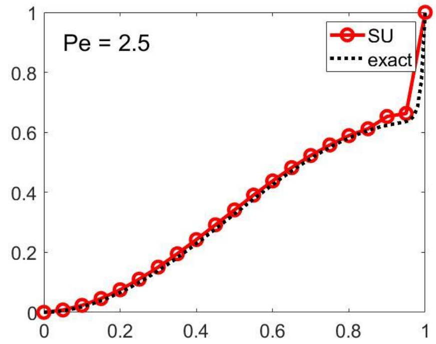


Stream upwind Petrov-Galerkin

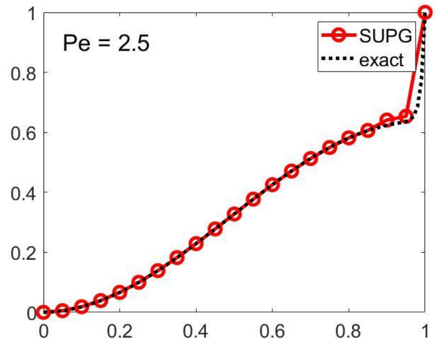


Galerkin/ Least-Squares

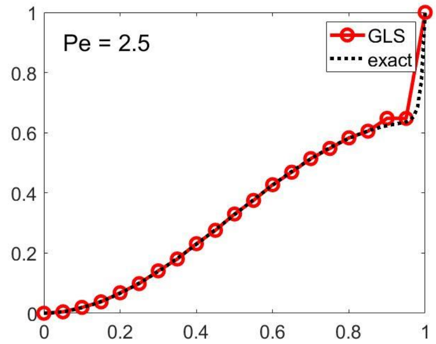
Exmaple3, $s = \sin(\pi x)$, $u_0 = 0$, $u_1 = 1$



Stream upwind

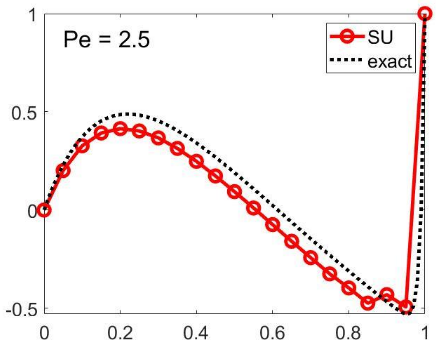


Stream upwind Petrov-Galerkin

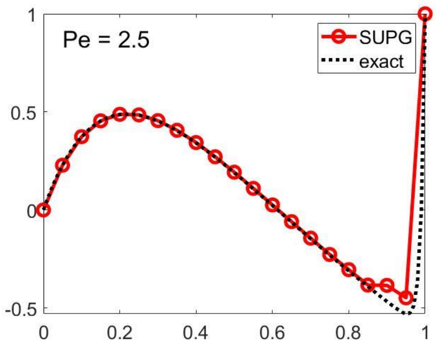


Galerkin/ Least-Squares

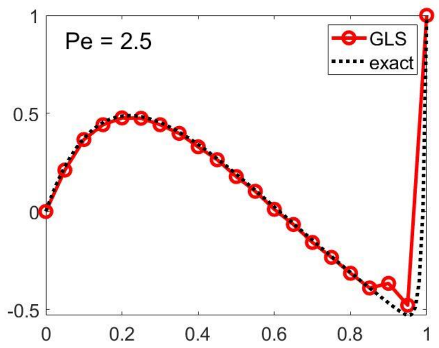
Exmaple4, $s = 10e^{-5x} - 4e^{-x}$, $u_0 = 0$, $u_1 = 1$



Stream upwind



Stream upwind Petrov-Galerkin



Galerkin/ Least-Squares

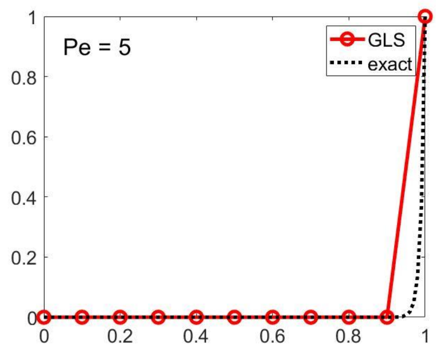
In quadratic elements, SU works better than in linear elements while the source term is variable, but it still worse than SUPG and GLS.

The SUPG and GLS performance in the same level. It will be changed while the second order derivative or diffusion term be added.

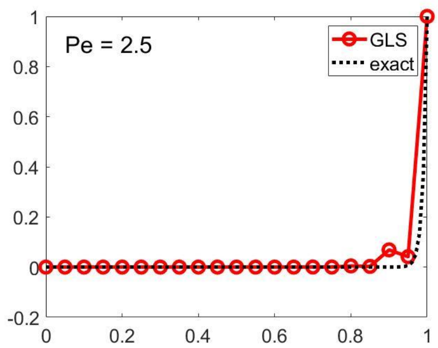
4. Conclusion

Compare with linear elements and quadratic elements, we can find that if the source term is constant or homogeneous, the linear elements is better choice than quadratic. But when the source term is variable, the quadratic is the better one.

Exmaple1, $s = 0, u_0 = 0, u_1 = 1,$

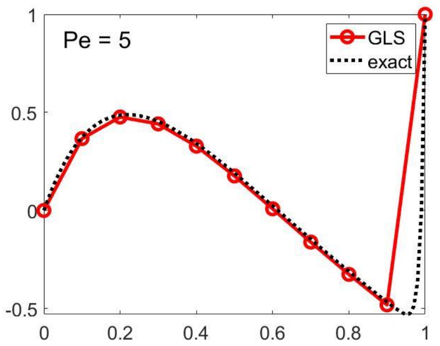


Galerkin/ Least-Squares linear element

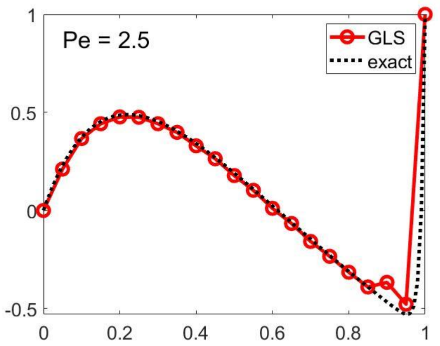


Galerkin/ Least-Squares quadratic element

Exmaple4, $s = 10e^{-5x} - 4e^{-x}$, $u_0 = 0$, $u_1 = 1$



Galerkin/ Least-Squares linear element



Galerkin/ Least-Squares quadratic element

5. Reference

[1] Lecture slides in Finite elements in fluid.