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MASTER OF SCIENCE IN COMPUTATIONAL MECHANICS

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**Finite Element in Fluids**  
Homework 4A - Unsteady Convection Problem

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# 1 UNSTEADY CONVECTION EQUATION IN 1D

## PROBLEM STATEMENT

Consider the convective transport of a quantity  $u$  which speed of propagation is given by a constant  $a$ , which is represented in the *convective* (or *advective*) form:

$$\begin{cases} u_t + au_x = 0 & x \in (0, 1), \quad t \in (0, 0.6) \\ u(x, 0) = u_0(x) & x \in (0, 1) \\ u(0, t) = 1 & t \in (0, 0.6] \end{cases} \quad (1.1)$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0.2, \\ 0 & \text{otherwise} \end{cases}$$

Where:

- $u$  - is the scalar unknown.
- $a = 1$  - is the convection velocity.
- $\Delta x = 0.02$  - is the spatial step.
- $\Delta t = 0.015$  - is the time step.

## COMPUTE THE COURANT NUMBER

The Courant Number is defined as:

$$C = \|a\| \frac{\Delta t}{h} \quad (1.2)$$

Substituting the values for the given problem:

$$C = \|1\| \frac{0.015}{0.02} = 0.75$$

SOLVE THE PROBLEM USING THE CRANK-NICOLSON SCHEME IN TIME AND LINEAR FINITE ELEMENT FOR THE GALERKIN SCHEME IN SPACE. IS THE SOLUTION ACCURATE?

The Crank-Nicolson scheme comes out from the discretization in time of the  $\theta$  methods, in which the value of  $\theta$  is 1/2. The general expression is:

$$\frac{\Delta u}{\Delta t} + \theta(a \cdot \nabla)u = \theta s^{n+1} + (1 - \theta)s^n - a \cdot \nabla u^n \quad (1.3)$$

After applying the weighted residual method and integrating by parts the corresponding terms which are necessary to impose continuity, the above equation can be expressed as:

$$\begin{aligned} & \left( w, \frac{\Delta u}{\Delta t} \right) - \theta (\nabla w, a \Delta u) + \theta ((a \cdot n) w, \Delta u)_{\Gamma^{out}} \\ & = (\nabla w, a u^n) - \theta ((a \cdot n) w, a u^n)_{\Gamma^{out}} + (w, \theta h^{n+1} + (1 - \theta) h^n)_{\Gamma_N^{in}} \\ & \quad + (w, \theta s^{n+1} + (1 - \theta) s^n) \end{aligned} \quad (1.4)$$

By considering the Galerkin approximation, this expression can be written in matrix form depending on the chosen shape functions:

$$\left( \frac{1}{\Delta t} \mathbf{M} + \theta \mathbf{C} \right) \Delta u = \mathbf{f} - \mathbf{C} u^n \quad (1.5)$$

Where:

**M:** is known as *Consistent Mass Matrix*

$$M_{ij} = \int_{\Omega} N_i N_j d\Omega \quad (1.6)$$

**C:** is the *Convective Matrix*

$$C_{ij} = \int_{\Omega} N_i (a \cdot \nabla N_j) d\Omega \quad (1.7)$$

**K:** is the *Stiffness Matrix*

$$C_{ij} = \int_{\Omega} \nabla N_i \nabla N_j d\Omega \quad (1.8)$$

Then, any type of approximation can be implemented as a shape function in the above equation. For this specific problem it is requested to use a linear approximation.

### Code Implementation 1

To perform the calculation by using the Matlab code provided in class, it is needed just to choose the initial condition problem (which is the option 4) and then give the correct parameters, which are convection velocity  $a = 1$ , number of elements that corresponds to  $nElem = 50$ , the end time  $tEnd = 0.6$  and number of time steps  $nSteps = 40$ . The resulting graphs is shown in figure 1.1

As it can be seen in the graph 1.1, Crank Nicolson scheme is not very accurate in the solution. The stability analysis provided by the theory mentions that as this is an implicit scheme and by using the Galerkin approximation, the amplification factor is always equal to one. Thus, *Crank-Nicolson is unconditionally stable and non dissipative*, but not provides accuracy.

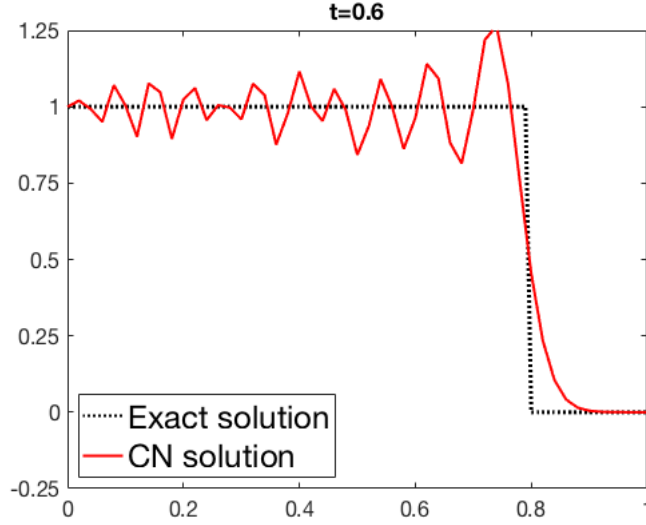


Figure 1.1: Solution graph using Crank-Nicolson time discretization and Galerkin spatial discretization.

SOLVE THE PROBLEM USING THE SECOND-ORDER LAX-WENDROFF METHOD. CAN WE EXPECT THE SOLUTION TO BE ACCURATE? IF NOT, WHAT CHANGES ARE NECESSARY? COMMENT THE RESULTS.

Lax-Wendroff (or TG2) methodology is an explicit second order accurate scheme, it is based upon a truncated Taylor series expansion, its time -stepping expression is:

$$\frac{\Delta u}{\Delta t} = -a \cdot \nabla u^n + \frac{\Delta t}{2} (a \cdot \nabla)^2 u^n + s^n + \frac{\Delta t}{2} (s_t^n - a \cdot \nabla s^n) \quad (1.9)$$

Then, the Galerkin spatial discretization can be done multiplying by a test function  $w$ , integrating over the domain, and then applying the integration by parts formula to obtain finally the next expression (in reduced notation):

$$\left( w, \frac{\Delta u}{\Delta t} \right) = \left( a \cdot \nabla w, u^n + \frac{\Delta t}{2} [s^n - (a \cdot \nabla) u^n] \right) - \left( (a \cdot n) w, u^n + \frac{\Delta t}{2} (a \cdot \nabla) u^n \right)_{\Gamma_{out}} + (w, h^{n+1/2})_{\Gamma_N^{in}} + \left( w, s^n + \frac{\Delta t}{2} s_t^n \right) \quad (1.10)$$

where:

$$h^{n+1/2} = - \left( u^n + \frac{\Delta t}{2} [s^n - (a \cdot \nabla) u^n] \right) (a \cdot n) \quad (1.11)$$

Then the matrix form of this scheme using the expressions 1.6, 1.7 and 1.8 is:

$$\frac{1}{\Delta t} \mathbf{M} \Delta u = \mathbf{f} + \mathbf{C} \mathbf{u}^n - \frac{\Delta t}{2} \|a\|^2 \mathbf{K} \mathbf{u}^n \quad (1.12)$$

### Code Implementation 2

To compute results using this scheme, the unique modification in the Matlab code was in the *System.m* file, and the matrices added to the case 1 are:

- $A = M;$
- $B = -dt*a*(C+0.5*dt*a*K);$
- $methodName = 'LW-G';$

Then, the code was ready to compute and the results are shown in the figure 1.2

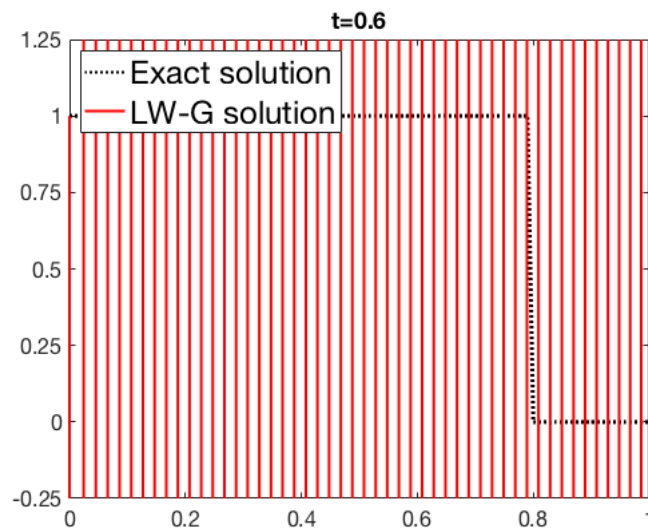


Figure 1.2: Solution graph using Lax-Wendroff time discretization and Galerkin spatial discretization.

As it can be seen the solution in the figure 1.2 provided by the Lax-Wendroff scheme is not accurate, and also includes tremendous spurious results which confirm that for Courant Number equal to 0.6, which is greater than stability limit  $C^2 < 1/3$  for this scheme, the solution “blows up” and is not useful.

In that sense, a different formulation based in Lax-Wendroff time scheme but employing a “Lumped Matrix” can be useful in order to reduce the oscillations caused by the instability of

the scheme. This formulation is based in transforming the *Mass Matrix* from a “consistent” formulation to a *Diagonal Mass Matrix*.

### Code Implementation 3

The implementation of the diagonalization of the mass matrix is done in the *FEM\_Matrices.m* file, in which the only modification is to multiply the mass matrix by a vector of ones (*one\_v*) as:

- $dMe = dMe + w\_ig * \text{diag}((N\_ig' * N\_ig) * \text{one\_v});$

Assembling, and modify the parameters of the function *System.m* to include this new diagonal matrix. Then the figure 1.3 shows the improved results.

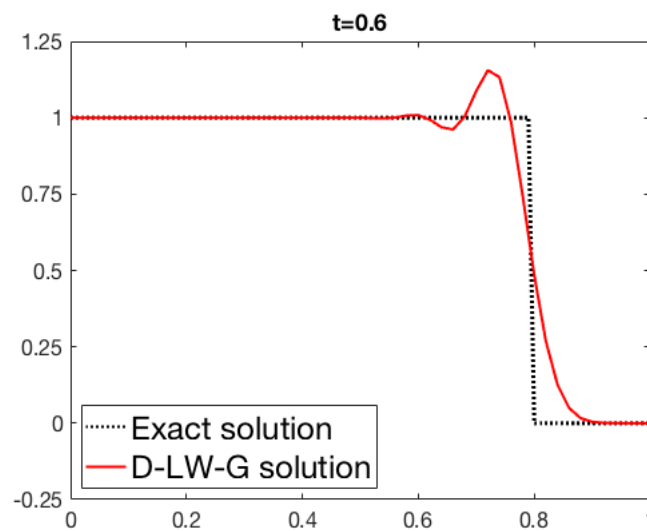


Figure 1.3: Solution graph using (Diagonal) Lax-Wendroff time discretization and Galerkin spatial discretization with Lumped Mass Matrix.

Using this modification in the mass matrix, there is a noticeable improvement in the Lax-Wendroff scheme. There is no oscillations and also the solution has a good accuracy.

SOLVE THE PROBLEM USING THE THIRD-ORDER EXPLICIT TAYLOR- GALERKIN METHOD. COMMENT THE RESULTS

The Third-Order Galerkin Method is based similarly than the Lax-Wendroff Scheme, but instead of using second accuracy in the truncation error of the Taylor series, TG3 truncates the third order. Thus, the time discretization is:

$$\begin{aligned} \left[1 - \frac{\Delta t^2}{6} (a \cdot \nabla)^2\right] \frac{\Delta u}{\Delta t} = & -a \cdot \nabla u^n + \frac{\Delta t}{2} (a \cdot \nabla)^2 u^n + s^n \\ & + \frac{\Delta t}{2} (s_t^n - a \cdot \nabla s^n) + \frac{\Delta t^2}{6} (s_{tt}^n - a \cdot \nabla s_t^n) \end{aligned} \quad (1.13)$$

Then, in order to approximate the solution, the weighted residual method is applied and integrated by parts as:

$$\begin{aligned} & \left(w, \frac{\Delta u}{\Delta t}\right) + \frac{\Delta t^2}{6} \left(a \cdot \nabla w, a \cdot \nabla \frac{\Delta u}{\Delta t}\right) - \frac{\Delta t^2}{6} \left((a \cdot n)w, a \cdot \nabla \frac{\Delta u}{\Delta t}\right)_{\Gamma_{out}} \\ & = \left(a \cdot \nabla w, u^n + \frac{\Delta t}{2} (a \cdot \nabla u^n)\right) - \left(a \cdot \nabla w, u^n + \frac{\Delta t}{2} (a \cdot \nabla u^n)\right)_{\Gamma_{out}} \\ & \quad + \frac{\Delta t}{2} \left(a \cdot \nabla w, s^{n+1/3}\right) - \frac{\Delta t}{2} \left((a \cdot n)w, s^{n+1/3}\right) \\ & \quad + \left(w, \frac{3}{4} s^{n+2/3} + \frac{1}{4} s^n\right) + \left(w, \frac{3}{4} h^{n+2/3} + \frac{1}{4} h^n\right)_{\Gamma_N^{in}} \end{aligned} \quad (1.14)$$

The matrix form can be written as:

$$\left(\frac{1}{\Delta t} \mathbf{M} + \frac{\Delta t}{6} \|a\|^2 \mathbf{K}u^n\right) \Delta u = \mathbf{f} + \mathbf{C}u^n - \frac{\Delta t}{2} \|a\|^2 \mathbf{K}u^n \quad (1.15)$$

#### Code Implementation 4

In order to include the TG3 method into the Matlab code, the lines added are:

- `A = M+(dt*dt/6)*a*a*K;`
- `B = -dt*a*(C+0.5*dt*a*K);`
- `methodName = 'TG3';`

The resulting graph is shown in the figure 1.4

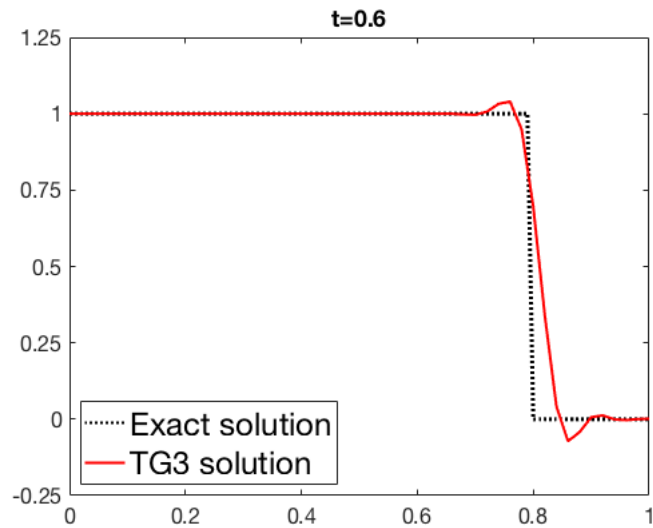


Figure 1.4: Solution graph using Third Order Taylor Galerkin Scheme.

The methodology of TG3 involves a completely better approximation by using third order accuracy in the time discretization. As can be seen in the figure 1.4, there is no oscillations or spurious values in the solution of this problem. The stability order now are all the Courant values under 1, that is why the discretization chosen works fine.