Finite Elements in Fluids **April 3, 2019**
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Assignment VII: Navier-Stokes problem.

Navier-Stokes equation:

$$\begin{cases} -\nu\nabla^2\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{b} & \text{in } \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{v}_D & \text{on } \Gamma_D \\ \mathbf{n}\cdot\sigma = \mathbf{t} & \text{on } \Gamma_N \end{cases}$$

Non-linear system of equations

$$\begin{pmatrix} \mathbf{K} + \mathbf{C}(\mathbf{v}) & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}$$

An iterative technique (Picard or Newton-Raphson methods) must be employed to iteratively solve the resulting system of nonlinear algebraic equations.

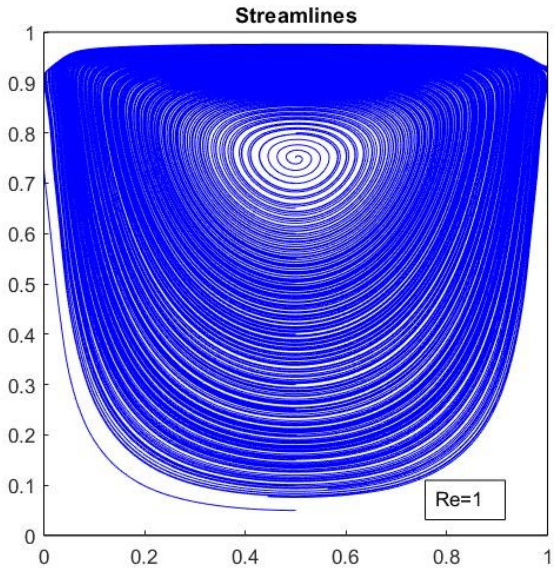
1 Convection matrix

$$[C(\mathbf{v^h})]_{ij} = \Big(\mathbf{N}_i, \Big(\Big(\sum_{j=1}^{n} \mathbf{v}_j N_j(\mathbf{x})\Big)\cdot\nabla\Big), \mathbf{v^h}\Big)$$

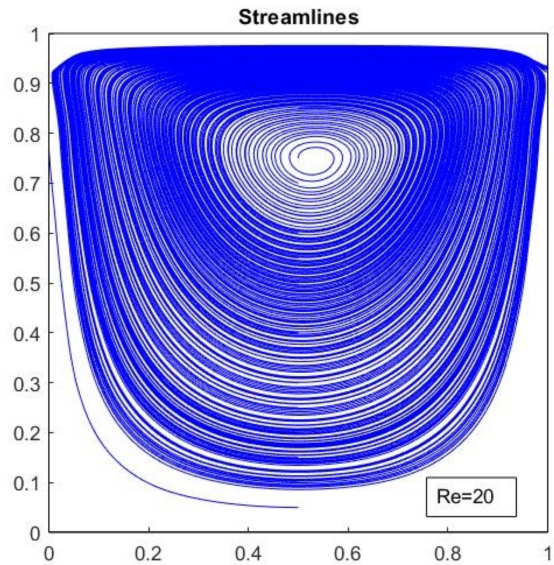
This term was implemented as follows in the code:

```
Ce = Ce + Ngp'*(v_igaus(1)*Nx+v_igaus(2)*Ny)*dvolu;
```

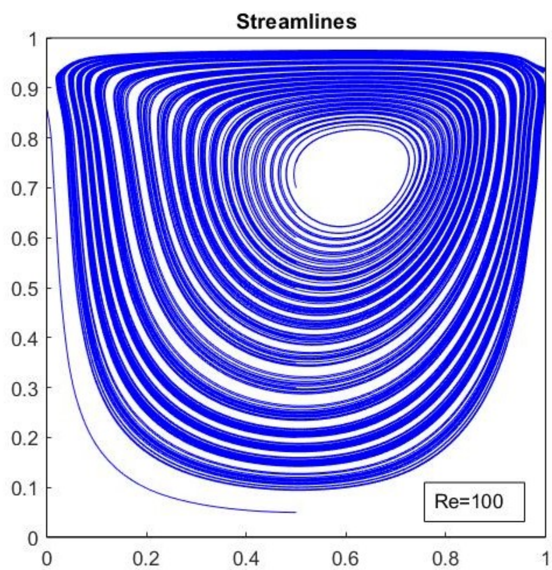
Using then Picard's method for different Reynolds numbers and for a number of elements of 10 in each direction we obtain the following results that can be observed in the following image:



(a)



(b)



(c)

Figure 1.1: Picard's method using Q2Q1 elements.

2 Newton-Raphson methods for non-linear system of equations

Newton-Raphson method to solve the Navier-Stokes non-linear system of equations. This method solves an equation of the form $\mathbf{r}(\mathbf{x}) = 0$ where, for the Navier-Stokes system, $\mathbf{r}(\mathbf{x})$ is defined as follows:

$$\mathbf{r} = \begin{bmatrix} \Big(\mathbf{K} + \mathbf{C}(\mathbf{v})\Big)\mathbf{v} + \mathbf{G}^T\mathbf{p} - \mathbf{f} \\ \mathbf{G}\mathbf{v} \end{bmatrix}$$

Thus, the solution after k iterations can be found by solving the linear system:

$$\begin{cases} \mathbf{J}(\mathbf{x}^k)\Delta\mathbf{x}^{k+1} = -r(\mathbf{x}^k) \\ \mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}^{k+1} \end{cases}$$

where $\mathbf{J}(\mathbf{x})$ is called the Jacobian matrix and it is mathematically defined as:

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial \mathbf{r}_1}{\partial \mathbf{v}} & \dfrac{\partial \mathbf{r}_1}{\partial \mathbf{p}} \\ \dfrac{\partial \mathbf{r}_2}{\partial \mathbf{v}} & \dfrac{\partial \mathbf{r}_2}{\partial \mathbf{p}} \end{pmatrix}$$

Where:

$$\frac{\partial \mathbf{r_1}}{\partial \mathbf{v}} = \mathbf{K} + \underbrace{\Big(\mathbf{N}_i, (\mathbf{v} \cdot \nabla), \mathbf{N}_j\Big)}_{\mathbf{C_2}(\mathbf{v})} + \underbrace{\Big(\mathbf{N}_i, (\mathbf{N}_j \cdot \nabla), \mathbf{v}\Big)}_{\mathbf{C_1}(\mathbf{v})}$$

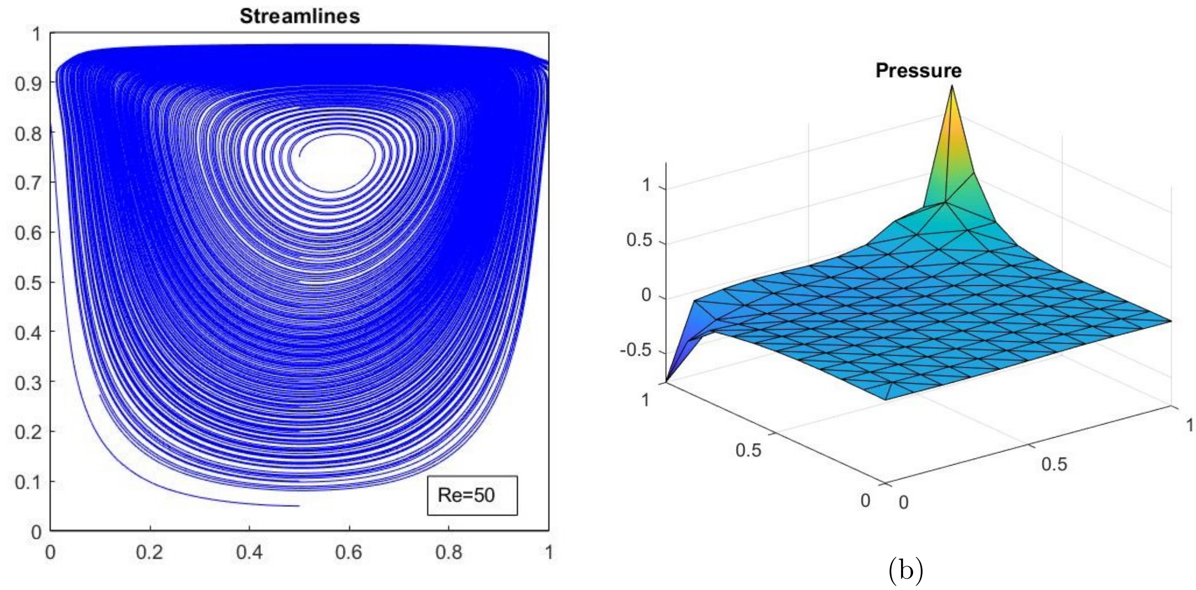
It is possible to see that $C_1(v) = C(v)$ from the Piccard's method but we have to discretize $C_2(v)$ as it follows:

$$\mathbf{w} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{w} \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} v_x & v_y \end{bmatrix}$$
$$= \mathbf{w} \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{bmatrix} \begin{bmatrix} v_x & v_y \end{bmatrix}$$

We will have then, the next convection terms:

```
Ce1 = Ce1 + Ngp'*(v_igaus(1)*Nx+v_igaus(2)*Ny)*dvolu;
Ce2 = Ce2 + Ngp'*([nx ; ny]*u_e)'*Ngp*dvolu;
```

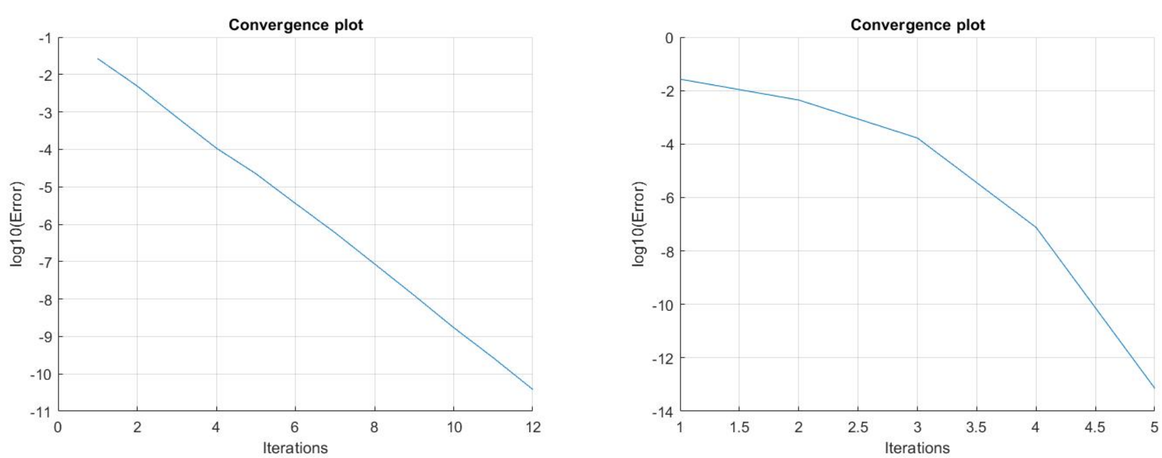
Finally, we can make the comparison of the two methods solving the problem for a Reynolds number of 50 and triangular elements P2P1:



(a)

(b)

Figure 2.1: Solution using Newton-Raphson Method



(a) Picard's method

(b) Newton-Raphson Method

Figure 2.2: Comparision of the convergence.