Finite Elements in Fluids **April 24, 2019**
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Assignment IX: UNSTEADY NAVIER STOKES PROBLEM

$$\begin{cases} v_t - \nu\nabla^2\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{b} & \text{in } \Omega x(0,T) \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega x(0,T) \\ \mathbf{v} = \mathbf{v}_D & \text{on } \Gamma_D x(0,T) \\ -p\mathbf{n}\cdot + \nu(\mathbf{n}\cdot\nabla)\mathbf{v} = \mathbf{t} & \text{on } \Gamma_N x(0,T) \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v_0}(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{1}$$

The weak formulation of system of equations (1) can be obtained by multiplying them by a vector weighting function w and a scalar weighting function q. The finite element discretization of the system of equations (1) is:

$$\begin{cases} \mathbf{M}\dot{\mathbf{u}}(t) + [\mathbf{K} + \mathbf{C}(\mathbf{v}(t))]\mathbf{u}(t) + \mathbf{G}\mathbf{p}(t) = \mathbf{f}(t, \mathbf{v}(t)) \\ \mathbf{G^T}\mathbf{u}(t) = \mathbf{h}(t) \\ \mathbf{u}(0) = \mathbf{v}_0 - \mathbf{v}_D(0) \end{cases} \tag{2}$$

where $\mathbf{M}$ is the standard finite element mass matrix defined as $\mathbf{M} = [mat\ N]^T[mat\ N]$.

1 Cavity flow problem:

The cavity flow problem is the reference point used to test the stabilization terms in this report. The geometry of the problem and the boundary conditions can be seen in the next figure:

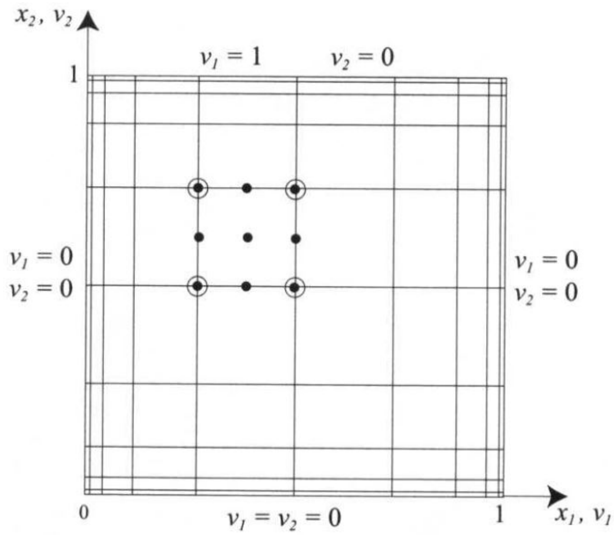


Figure 1.1: Problem statement with boundary conditions.

2 Chorin-Temam projection method

The principle of the projection method is to compute the velocity and pressure fields separately through the computation of an intermediate velocity, which is then projected onto the subspace of the solenoidal vector functions.
The Chorin-Temam projection method includes two basic steps as follows.

1. includes the viscous and convective in convective terms in the Navier-Stokes equations and (given the previous velocity field v^n) consist of finding an intermediate velocity field v^{n+1} such that:

$$\begin{cases} \dfrac{\mathbf{v}_{int}^{n+1} - \mathbf{v}^n}{\Delta t} + (\mathbf{v}^* \cdot \nabla)\mathbf{v}^{**} - \nu\nabla^2\mathbf{v}^{**} = \mathbf{f}^{n+1} & \text{in } \Omega \\ \mathbf{v}_{int}^{n+1} = \mathbf{v}_D^{n+1} & \text{on } \Gamma \end{cases} \tag{3}$$

   where $\mathbf{v}^*$ and $\mathbf{v}^{**}$ must be chosen suitable for the treatment of the nonlinear convective term, possible options are:

   - Explicit Euler $\mathbf{v}^*$=$\mathbf{v}^{**}$=$\mathbf{v}^n$

- Semi-Implicit $\mathbf{v}^{*}$=$\mathbf{v}^{n}$ and $\mathbf{v}^{**}$=$\mathbf{v}_{int}^{n+1}$
- Implicit Euler $\mathbf{v}^{*}$=$\mathbf{v}^{**}$=$\mathbf{v}_{int}^{n+1}$

For the semi-implicit and fully implicit cases, the algebraic system resulting from the finite element discretization is:

$$\mathbf{M}_1 \left(\frac{\mathbf{v}_{int}^{n+1} - \mathbf{v}^{n}}{\Delta t} \right) + (\mathbf{C}(\mathbf{v}^{*}) + \mathbf{K})\mathbf{v}_{\mathbf{int}}^{\mathbf{n+1}} = \mathbf{f}^{n+1} \tag{4}$$

2. determines the velocity $\mathbf{v}^{\mathbf{n+1}}$ and pressure $\mathbf{p}^{\mathbf{n+1}}$ solving:

$$\begin{cases} \dfrac{\mathbf{v}_{int}^{n+1} - \mathbf{v}^{n}}{\Delta t} + \nabla p^{n+1} = 0 & \text{in } \Omega \\ \nabla \cdot v^{n+1} = 0 & \text{in } \Omega \\ \mathbf{n} \cdot \mathbf{v}_{int}^{n+1} = \mathbf{n} \cdot \mathbf{v}_{D}^{n+1} & \text{on } \Gamma \end{cases} \tag{5}$$

The discrete equations emanating from the discretization of the weak form of the above equations induce the following system of algebraic equations:

$$\begin{cases} \mathbf{M}_2 \left(\dfrac{\mathbf{v}^{n+1} - \mathbf{v}_{int}^{n+1}}{\Delta t} \right) + \mathbf{G}\mathbf{p}^{n+1} = 0 \\ \mathbf{G}^{\mathbf{T}}\mathbf{v}^{n+1} = 0 \end{cases} \tag{6}$$

Code Implementation:

```
while step < nstep
step = step +1;
C = ConvectionMatrix (X,T, referenceElement , velo );
Cred = C(dofUnk , dofUnk );
fredn = fred - (K(dofUnk , dofDir )+C(dofUnk , dofDir ))* valDir ;

% FIRST STEP
btot = dt* fredn + Mred * veloVect ( dofUnk );
Atot = Mred +dt *( Cred + Kred );
Z = Atot \ btot ;

% SECOND STEP
btot = [ Mred *Z; f_q ];
Atot = [ Mred Gred '* dt; Gred L];
aux = Atot \ btot ;

veloInc = zeros (ndofV ,1);
veloInc ( dofUnk ) = aux (1: nunkV );
presInc = aux ( nunkV +1: end );
```

```
    velo = reshape ( veloInc ,2 ,[]) ';
    pres = presInc ;
    end
```

3 Semi-implicit first order monolitic scheme

$$\frac{\Delta\mathbf{u}}{\Delta t} - \theta\Delta\mathbf{u}_t = \mathbf{u}_t^n \tag{7}$$

where the θ is in the interval [0,1] and determines which method is apply. When $\theta \geq 0.5$,the methods are unconditionally stable.
Consider the transient weak form of Navier-Stokes:

$$\mathbf{u}_t = \frac{\mathbf{f} - (\mathbf{K} + \mathbf{C}(v))\mathbf{u} - \mathbf{G}\mathbf{p}}{M} \tag{8}$$

Substituting this equation in the previous equation:

$$\frac{\Delta\mathbf{u}}{\Delta t} - \theta\left[\left(\frac{-(\mathbf{K} + \mathbf{C^{n+1}})\mathbf{u}^{n+1} - \mathbf{G}\mathbf{p}^{n+1}}{M}\right) - \left(\frac{-(\mathbf{K} + \mathbf{C^{n}})\mathbf{u}^{n} - \mathbf{G}\mathbf{p}^{n}}{M}\right)\right] = \frac{\mathbf{f} - (\mathbf{K} + \mathbf{C^{n}})\mathbf{u}^{n} - \mathbf{G}\mathbf{p}^{n}}{M} \tag{9}$$

The nonlinear convective term $\mathbf{C}$ is considered as linearized, and the above equation can be re-arranged as:

$$\frac{\Delta\mathbf{u}}{\Delta t} + \theta\left(\frac{(\mathbf{K} + \mathbf{C^{n}})\Delta\mathbf{u} + \mathbf{G}\Delta\mathbf{p}}{M}\right) = \frac{\mathbf{f} - (\mathbf{K} + \mathbf{C^{n}})\mathbf{u}^{n} - \mathbf{G}\mathbf{p}^{n}}{M} \tag{10}$$

To complete the above system, the divergence free condition must be imposed. Then the equations can be written as:

$$\begin{cases} (\mathbf{M} + \theta\Delta t(\mathbf{K} + \mathbf{C}^n))\Delta\mathbf{u} + \theta\Delta t\mathbf{G}\Delta\mathbf{p} = \Delta t(\mathbf{f} - (\mathbf{K} + \mathbf{C}^n)\mathbf{u}^n - \mathbf{G}\mathbf{p}^n) \\ \mathbf{G}^T\Delta\mathbf{u} = 0 \end{cases} \tag{11}$$

The solution of the transient Navier-Stokes problem can be found using the time discretization by the θ methods. In order to implement this formulation into the Matlab code, first consider the equation (11) in matrix form as:

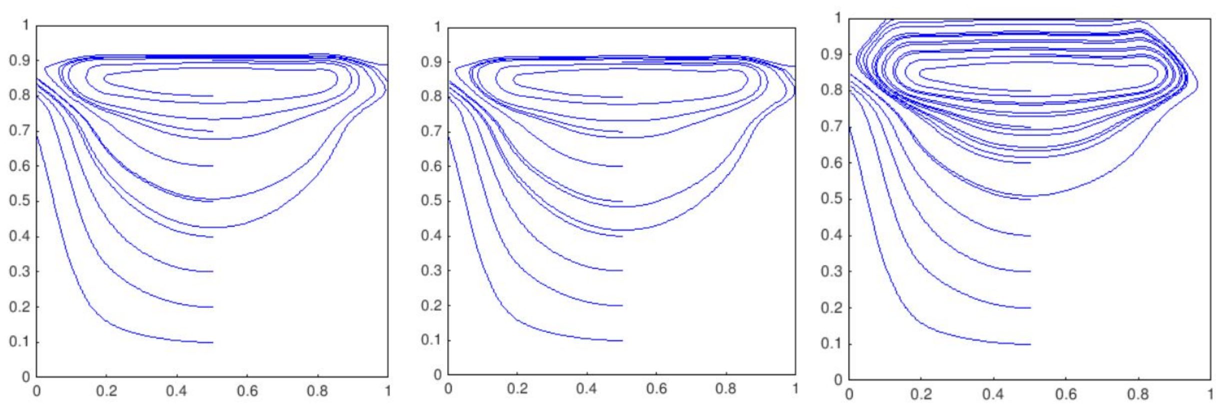
$$\begin{bmatrix} \mathbf{M} + \theta\Delta t(\mathbf{K} + \mathbf{C}^n) & \theta\Delta t\mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{p} \end{bmatrix} = \begin{bmatrix} \Delta t(\mathbf{f} - (\mathbf{K} + \mathbf{C}^n)\mathbf{u}^n - \mathbf{G}\mathbf{p}^n) \\ \mathbf{0} \end{bmatrix} \tag{12}$$

Code Implementation:

```
while step < nstep
    step = step +1;
    C = ConvectionMatrix (X,T, referenceElement , velo );
    Cred = C(dofUnk , dofUnk );
    fredn = fred - (K(dofUnk , dofDir )+C(dofUnk , dofDir ))* valDir ;

    % Matricial system of equations
    Atot = [ Mred + teta *dt *( Kred + Cred ) dt* teta *Gred'
    Gred zeros ( nunkP )];
    btot = [dt *( fredn -( Kred + Cred )* veloVect ( dofUnk )-Gred'* ...
        pres );zeros (nunkP ,1)];

    % Computation of velocity and pressure increment
    solInc = Atot \ btot ;

    % Update of the solution
    veloInc = zeros (ndofV ,1);
    veloInc ( dofUnk ) = solInc (1: nunkV );
    presInc = solInc ( nunkV +1:  end );
    velo = velo + reshape ( veloInc ,2 ,[])';
    pres = pres + presInc ;
end
```

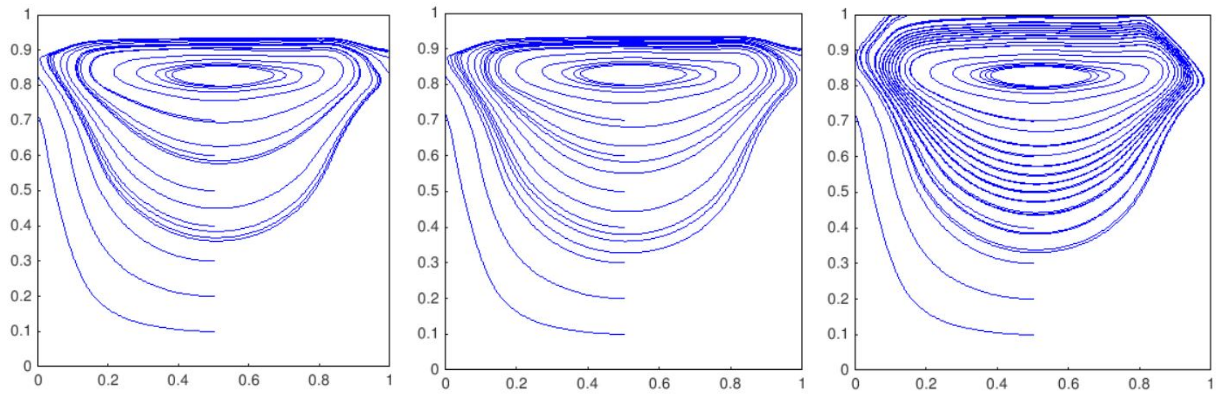
4 Results

The domain was discretized in a regular 5x5 mesh of Q2Q1 elements.
In the following plots the three methods used will be shown for different time steps:



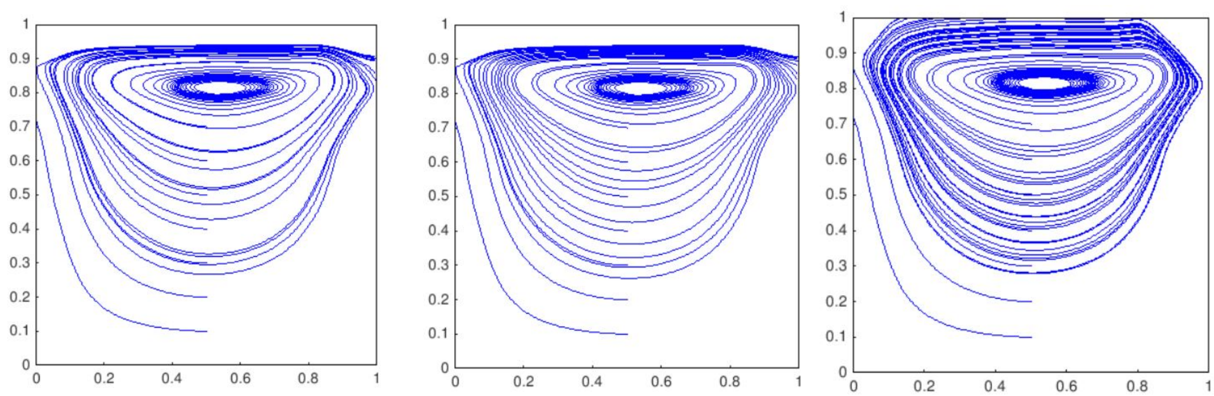
(a) $\theta = 1$ (b) $\theta = 0.5$ (c) Chorin-Temam

Figure 4.1: Time step 10



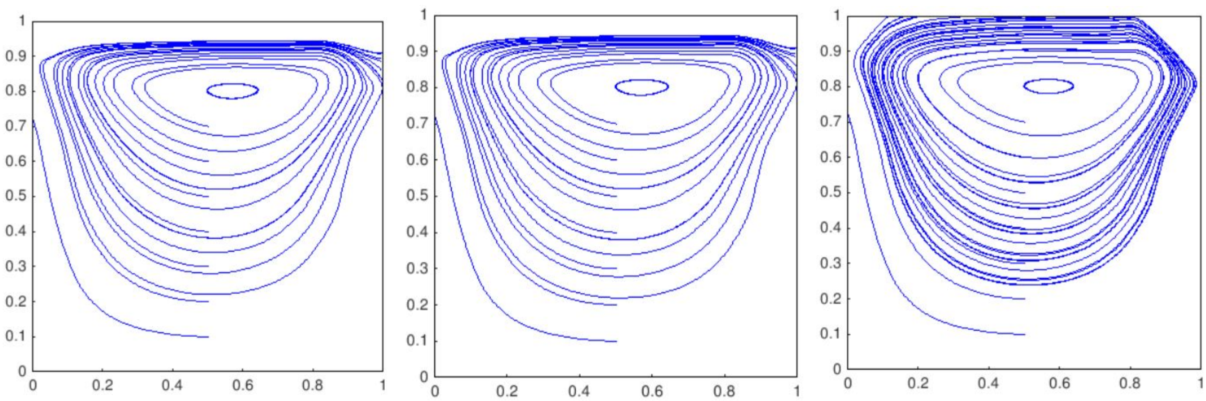
(a) $\theta = 1$ (b) $\theta = 0.5$ (c) Chorin-Temam

Figure 4.2: Time step 30



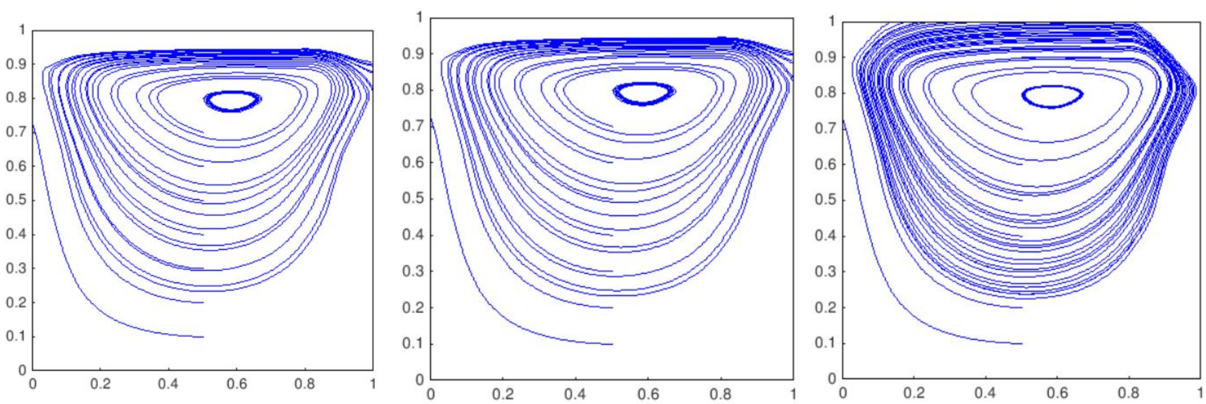
(a) $\theta = 1$ (b) $\theta = 0.5$ (c) Chorin-Temam

Figure 4.3: Time step 50



(a) $\theta = 1$ (b) $\theta = 0.5$ (c) Chorin-Temam

Figure 4.4: Time step 80



(a) $\theta = 1$ (b) $\theta = 0.5$ (c) Chorin-Temam

Figure 4.5: Time step 100

The graphics show that the calculations are very similar for all time steps. Some differences can be observed in the density of the flow lines.

The most notorious difference is observed between Chorin-Temam and the semi-implicit methods, the upper position of the flow. (Chorin-Temam predicts that it will be larger).