IMPLICIT FIRST ORDER MONOLITIC SCHEME

Used scheme:

$$\frac{v^{n+1} - v^n}{\Delta t} + C(v) + K(v^{n+1}) + \nabla p^{n+1} = f^{n+1}$$

$$C(v)^{n+1} = (v^{n+1} \cdot \nabla)\, v^{n+1} \qquad \Big\| \qquad \nabla \cdot v^{n+1} = 0$$

$$K(v^{n+1}) = -\nu \nabla^2 v^{n+1}$$

WEAK FORM: (to compute $n+1 \Rightarrow$ results for n known)

Consider $u \in V$ s.t. $u = 0$ on Γ_D

$q \in Q$

$$\int_\Omega u \left[\frac{v^{n+1} - v^n}{\Delta t} + C(v) + K(v^{n+1}) + \nabla p^{n+1} \right] d\Omega = \int_\Omega u f^{n+1} d\Omega$$

$$\int_\Omega q\, \nabla \cdot v^{n+1} d\Omega = 0.$$

- Applying $\nabla \cdot (u \cdot \nabla v) = u \cdot \nabla^2 v + \nabla u : \nabla v$. and the divergence theorem

$$\int_\Omega u\, K(v^{n+1}) d\Omega = -\int_\Omega u \cdot \nu \nabla^2 v^{n+1} d\Omega = \int_\Omega \nabla u : \nu \nabla v^{n+1} d\Omega - \int_{\partial\Omega} u \cdot \nu \nabla v^{n+1} \cdot n \, d\Gamma$$

$$= \int_\Omega \nabla u : \nu \nabla v^{n+1} d\Omega - \cancel{\nu \int_{\Gamma_D} \overset{0}{u}\, \nabla v^{n+1} \cdot n \, d\Gamma} - \nu \int_{\Gamma_N} u\, t \, d\Gamma$$

- Applying $\nabla(u \cdot p) = (\nabla u) \cdot p + u \nabla p$ an the divergence theorem.

$$\int_\Omega u \nabla p^{n+1} d\Omega = -\int_\Omega (\nabla \cdot u) \cdot p^{n+1} d\Omega + \int_\Gamma u \cdot p \cdot n \, d\Gamma = -\int_\Omega (\nabla \cdot u) \cdot p^{n+1} d\Omega$$

So we end with:

$$\boxed{\begin{gathered} \int_\Omega \left[\frac{1}{\Delta t} u\, v^{n+1} + u\,(v^n \cdot \nabla)\, v^{n+1} + \nu \nabla u : \nabla v^{n+1} - (\nabla \cdot u)\, p^{n+1} \right] d\Omega = \\ = \int_\Omega \left[u f^{n+1} + \frac{1}{\Delta t} u\, v^n \right] d\Omega + \nu \int_{\Gamma_N} u\, t \, d\Gamma \\ \int q\, \nabla \cdot v^{n+1} d\Omega = 0 \end{gathered}}$$

Using a FE Galerkin discretization ; where m is the dimension of the domain

$$v^n \approx (v^h)^n = [\text{mat } N]\, V^n \quad , \quad \begin{bmatrix} \frac{\partial N_1^n}{\partial x_1} \\ \vdots \\ \frac{\partial N_m^n}{\partial x_1} \\ \frac{\partial N_1^n}{\partial x_2} \\ \vdots \\ \frac{\partial N_n^n}{\partial x_m} \end{bmatrix} = g\left((v^h)^n\right) = [\text{grad } N]\, V^n$$

and $\nabla \cdot v^h = [1,0,0,1]\,[\text{grad } N]\, V^n = D V^n$

in 2D ↗ or $[100010001]$

in 3D ↗

(same for u)

$$P^n \approx (p^h)^n = \hat{N} P^n \qquad \text{(same for } q\text{)}$$

Imposing the discretization on the weak form :

$$(1)\ \int_\Omega \Big[\tfrac{1}{\Delta t}\big([\text{mat } N]U\big)^T\big([\text{mat } N]V^{n+1}\big) + \big([\text{mat } N]U\big)^T (v^{n+1}) \cdot \big([\text{grad } N]\, V^{n+1}\big)$$

$$+\nu\big([\text{grad } N]U\big)^T\big([\text{grad } N]V^{n+1}\big) - (D\,U)^T[\hat{N}]P^{n+1} \Big]\, d\Omega \;\Leftrightarrow$$

$$\Leftrightarrow \int_\Omega \Big(\big([\text{mat } N]U\big)^T f^{n+1} + \tfrac{1}{\Delta t}\big([\text{mat } N]U\big)^T\big([\text{mat } N]V^n\big)\Big)\, d\Omega$$

$$+\nu\int_{\Gamma_N} \big([\text{mat } N]U\big)^T t \; d\Gamma$$

$$(2)\ \int_\Omega \big([\hat{N}]Q\big)^T D V^{n+1}\, d\Omega = \int_\Omega \big([\hat{N}]Q\big)^T 0 \; d\Omega$$

$$\Downarrow$$

$$(1)\ U^T\Big[\Big[\underbrace{\tfrac{1}{\Delta t}\int_\Omega [\text{mat } N]^T[\text{mat } N]\, d\Omega}_{M} + \underbrace{\int_\Omega [\text{mat } N]^T (v^{n+1})\,[\text{grad } N]}_{C(v^{n+1})}$$

$$+ \underbrace{\nu\int_\Omega [\text{grad } N]^T[\text{grad } N]\, d\Omega}_{K}\Big] V^{n+1} - \Big[\underbrace{\int_\Omega D^T[\hat{N}]\, d\Omega}_{-G^T}\Big] P^{n+1}\Big] \Leftrightarrow$$

$$\cancel{\Leftrightarrow Q^T\Big[\underbrace{\int_\Omega [\hat{N}]^T D\, d\Omega}_{G}\Big] V^{n+1} = Q^T\Big[\underbrace{\int_\Omega [\hat{N}]^T 0\, d\Omega}_{0}\Big]}$$

$$\Leftrightarrow U^T\Big[\Big[\underbrace{\tfrac{1}{\Delta t}\int_\Omega [\text{mat } N]^T[\text{mat } N]\, d\Omega}_{M}\Big] V^n + \underbrace{\int_\Omega [\text{mat } N]^T f^{n+1} d\Omega}_{f} + \underbrace{\nu\int_{\Gamma_N} [\text{mat } N]^T t\, d\Gamma}_{t}\Big]$$

(2) $Q^T \Big[\underbrace{\int_\Omega [\hat{N}]^T D \, d\Omega}_{G} \Big] V^{n+1} = Q^T \underbrace{\int_\Omega [\hat{N}] \cdot 0 \, d\Omega}_{\overline{0}}$

Since the weak form has to be acomplished $\forall\, u, q$ we, end with the nonlinear system (and we can consider the change of variables $p = -p$)

$$\begin{cases} \left[\frac{1}{\Delta t} M + C(v^{n+1}) + \nu K \right] V^{n+1} + G^T p^{n+1} = \frac{1}{\Delta t} M V^n + \vec{f} + \nu \vec{t} \\ G \, V^{n+1} = \overline{0} \end{cases}$$

We can compute M using

$M = \underset{e}{\mathrm{A}} \, Me \quad \leftarrow \quad M(Te.dof, Te.dof) = M(Te.dof, Te.dof) + Me$

where

$Me = Me + Ngp' * Ngp * dvolu;$ $\forall ig$ ~~int~~ gauss point of the e-th element.

I can not plot the results since I do not know what initial conditions impose to the PDE and I do not know the time interval that we are considering