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Stochastic Finite Element Modeling and Its Applications in Heat Conduction and Elastostatics Problems

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Summary

In the design of engineering structures, the heterogeneous materials such as soils, rocks, steel, concrete, etc. are usually assumed to take constant material properties. Deterministic mathematical models are used to predict the approximate natural responses of these engineering structures. As a result, conservative factors of safety are introduced in the design using the mean-value solution which gives a significant increase in the design, construction and operational costs.

In reality, all heterogeneous materials have inherent uncertainties. These uncertainties of materials can be modeled mathematically leading to Stochastic Partial Differential Equations (SPDEs) and can be solved using Stochastic Finite Element Method (SFEM). The stochastic models will provide realistic simulations of physical phenomena and give the analyst specific information on the probabilities that can be assigned to predictions.

The aim of this thesis is to study numerical techniques in SFEMs and develop applications in heat conduction and elastostatics problems. The task involves material modeling of random media, numerical solver of stochastic linear systems and applications in heat conduction and elastostatics problems. In this thesis, the random material properties are modeled as stochastic fields and discretized using the Fourier-Karhunen-Loève representation scheme. Then the joint diagonalization solution strategy is investigated for the solution of the resulting stochastic linear systems and attempts are made to improve the performance with joint tridiagonalization of multiple matrices. These numerical techniques are integrated into a SFEM framework, which is then tested in heat conduction and elastostatics problems of concrete wastewater pipe.

In conclusion, the joint diagonalization algorithm agrees well with the results of Monte Carlo method and Neumann expansion method. Also, numerical results confirmed that random material properties have significant effects on the structural responses of the concrete pipe. Moreover, the joint tridiagonalization strategy is also being developed but due to time constraint some problems in convergence are still to be worked out.

Declarations and Statements

(i) This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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(ii) This dissertation is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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In memory of my father

ELMO C. AQUINO SR.

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Chapter 1

Introduction

1.1 Background and Motivation

"Randomness is the lack of pattern or regularity" [1.1]. The two sources of randomness [1.2] are, (i) the impracticality of a comprehensive deterministic description and the inherent irregularity in the phenomenon being observed, and (ii) the generalized lack of knowledge about the processes concerned. The level of uncertainty in the second class of problem, typically can be lessened by improving the measuring tools through which the process is being monitored and by properly documenting more data of the process. In the design of engineering structures, materials with random properties called random medium are often used, such as, but not limited to soils, rocks, and concrete. However, the phenomenon involving these random media is not only encountered in civil engineering but also in almost every engineering sector. Typical engineering problems as shown in Figure 1.1 with respect to random media include: (a) the analysis of contamination of soil and groundwater with spatially distributed random hydro-geological fields (e.g. hydraulic conductivity); (b) the mechanics of soft and hard tissues with different constitutive models; (c) the electromagnetic waves scattering by a perfectly conducting aircraft; (d) risk assessment of rock structures; (e) the design of civil engineering structures under static, dynamic (earthquake and wind loadings), random fluctuations in temperature, humidity,

and other environmental loading conditions; (f) life prediction of structures/bridges, etc., all of which require complex mathematical predictions. The natural responses of these structures may be completely unpredictable by deterministic models.

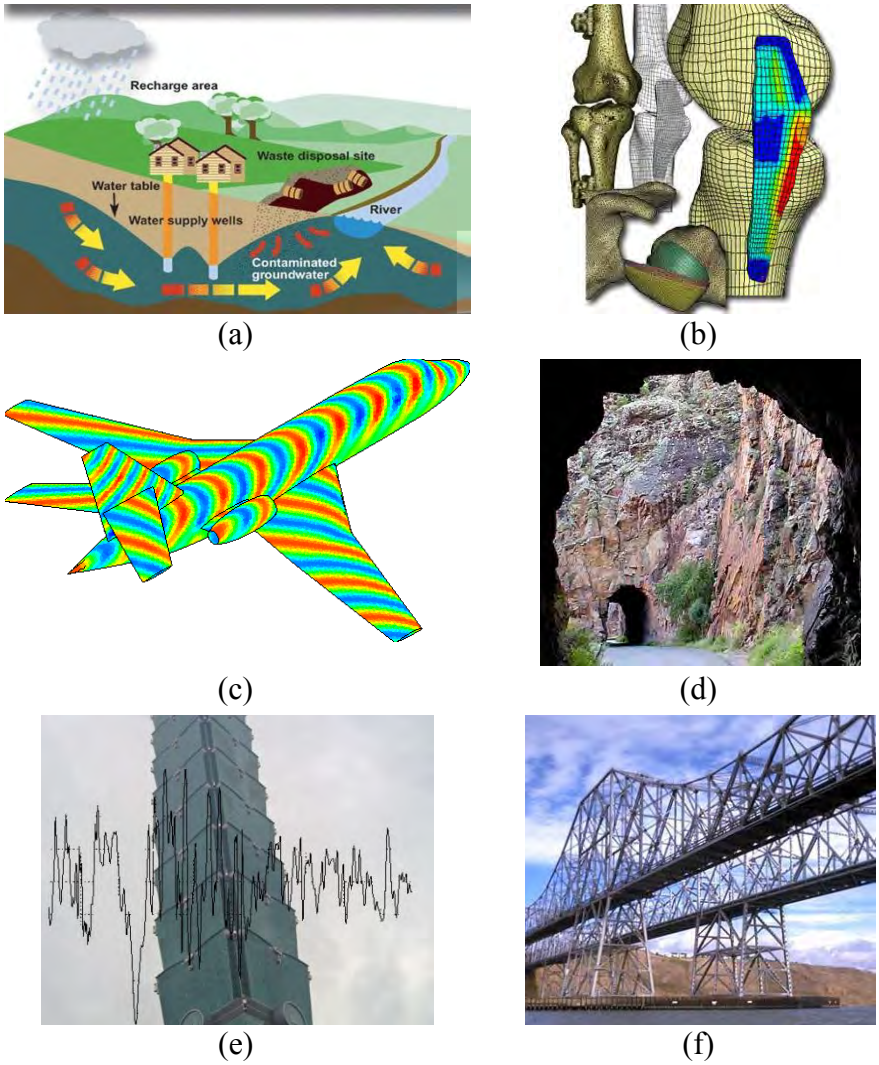

Figure 1.1 Engineering systems/structures with variable uncertainties

Fortunately, these uncertainties and random characteristics of materials can be modeled mathematically using continuum mechanics which provides comprehensive mathematical models in the form of partial differential equations. These probabilistic or stochastic models will not only provide realistic simulations of physical phenomena but will also provide the analyst with specific information on the probabilities that can be assigned to predictions. Nevertheless, the understanding of engineering structures with variable uncertainties is not

yet fully developed. This is due to the fact that the stochastic nature of random media is so complex, the lack of proper mathematical model for the related stochastic fields, and the inadequacy of supporting analysis tools for the corresponding Stochastic Partial Differential Equations (SPDEs).

Today, deterministic mathematical models are still widely used in the analysis of engineering structures composed of random media. This result to the introduction of the conservative factors of safety applied to the approximate mean-value solution, which gives a significant increase in the design, construction and operational costs. Apparently, deterministic models can be considered only as approximations to the corresponding physical problems. Li [1.3] has enumerated the four major disadvantages of using deterministic mathematical models to study random media. The lack of a versatile stochastic model for random media and the inadequacy of compatible analysis tools for the corresponding SPDEs are the reasons why deterministic models are still widely used. Therefore, it is of vital importance to investigate the mathematical foundation and to develop efficient and robust algorithms for practical engineering structures consisting of random media.

The random structural response of engineering structures can be stimulated either by the input or the system operator [1.1]. In computational structural mechanics, different types of problems will arise based on what information is available as shown in Table 1.1 [1.4]. From an engineering mechanics point of view, the most common stochastic system problem involves a linear differential equation with random coefficients [1.1]. In which the properties of the system under study are represented by these coefficients and can be interpreted as random variables. The problem can be written mathematically as

$$\boldsymbol{\Lambda u} = \boldsymbol{f} \tag{1.1}$$

where $\boldsymbol{\Lambda}$ is a linear stochastic differential operator, $\boldsymbol{u}$ is the random response, and $\boldsymbol{f}$ is possibly deterministic/random excitation. This present research study undertakes only the deterministic input and random system type of problems. Hence, Stochastic Finite Element Method (SFEM) will be used (see Table 1.1).

Input	System	Output	Problem Name	Main Techniques
Known[a]	Known[a]	Unknown	Analysis (Forward Prob.)	FEM/BEM/FD
Known[a]	Incorrect[a]	Known[a]	Updating/Calibration	Modal Updating
Known[a]	Unknown	Known[a]	System Identification	Kalman Filter
Assumed[a]	Unknown[a]	Prescribed	Design	Design Optimization
Known	Partially Known	Known	Structural Health Monitoring (SHM)	SHM Methods
Known[a]	Known[a]	Prescribed	Control	Modal Control
Known[b]	Known[a]	Unknown	Random Vibration	Random Vibration
Known[a]	Known[b]	Unknown	Uncertainty Propagation (Forward Problem)	SFEM/SEA/RMT
Known[c]	Known[b]	Known from experiment and model[b]	Model Validation	Validation Methods
Known[c]	Known[b]	Known from different computations[b]	Model Verification	Verification Methods
Known[b]	Incorrect[b]	Known[b]	Probabilistic Updating/Calibration	Bayesian Calibration
Assumed[c]	Unknown[b]	Prescribed[b]	Probabilistic Design	RBOD
Known[c]	Partially Known[b]	Partially Known[b]	Joint State and Parameter Estimation	Particle Kalman Filter/ Ensemble Kalman

Note: [a] Deterministic, [b] Random, [c] Random/Deterministic

Table 1.1 Types of problems in Computational Structural Mechanics

1.2 Review of Available Techniques

Mathematicians and engineers study and discover independently the theoretical and/or numerical aspects of random medium mechanics from different points of view, especially in the fields of SPDEs and SFEMs, respectively. In this perspective, several methods have been developed and proposed in these fields of study. Most of the available methods in SFEMs and its development will be presented next.

1.2.1 Stochastic Partial Differential Equations

The theory of Stochastic Ordinary Differential Equations (SODEs) also known as the Itô integral [1.5], have been established in 1980s and its successful applications have been observed in a wide range of scientific and/or technical areas. Following the triumph of SODEs, researchers have attempted to extend Itô's white noise model from one dimension into higher dimensions [1.6]. Unfortunately, there is no natural total-order structure in

higher dimensional space. However, engineers believe that these SPDE developments are physically relevant, even though some mathematical results acquired in SPDEs show that it has little to do with specific physical problems and are useful only for their own interest. Accordingly, an effective SPDE theory is still not available to account for practical engineering structures composed of random media.

1.2.2 Stochastic Finite Element Methods

The Finite Element Method (FEM) has become the dominant analysis tool in engineering since over the last few decades. All the parameters in the standard FEM assumed constant values. In the viewpoint of discovering the appropriate method to solve stochastic systems, engineers tried to expand the standard FEM into SFEM [1.17-1.46] basically by substituting some of the parameters in the FEM by random variables. The important developments in SFEM related research are summarized as follows:

Shinozuka [1.7] applied the Monte Carlo method and the standard FEM in the reliability analysis of structures with random excitation, random material properties or random geometric configurations [1.18-1.21]. The stochastic computational mechanics, also known as computational stochastic mechanics, then appeared from the related research area undertaken and a substantial growth emerged. In this work, Shinozuka used the trigonometric series approximation method, which is based on the spectral representation theory of wide-sense stationary stochastic fields [1.17, 1.22-1.24]. He also used the First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) [1.25-1.26] techniques. Rackwitz [1.28] noted that FORM and SORM are the most popular approximate techniques to date, which are used in calculating the probability integral encountered in reliability analysis of random structures. Furthermore, one of the latest research interests of these methods is applying them to larger structures [1.27].

Vanmarcke [1.8], Kiureghian [1.9], and Grigoriu [1.10] are also the pioneering researchers in stochastic computational mechanics. During the period of 1980s and the 1990s, more researchers joined in the research of SFEMs. Among the distinguished researchers are Liu [1.11], Spanos [1.12], Ghanem [1.1, 1.13], Kleiber [1.16], Deodatis

[1.14], and Schuëller [1.15]. Many interesting results during this era were developed especially a number of non-Monte Carlo SFEM formulations.

Perhaps, the first SFEM formulation which is not based on Monte Carlo simulations was done by Vanmarcke and Grigoriu [1.29]. The local averaging representation scheme for random material properties of beam elements is developed. The method is limited to estimates of second-order statistical quantities, i.e. the expectation and covariance.

Lawrence [1.32] developed the SFEM formulation based on the series expansion and the Galerkin method. The equation of the form $Ku = f$ was considered. The terms K, u, and f are expanded into finite series consisting of random coefficients and deterministic base functions, then Galerkin approach was used to solve the equation.

Liu, et al. [1.30-1.31] developed the perturbation method. This method starts by expanding the unknown stochastic field using Taylor's expansion. The stochastic field of random media is represented by random variables in which Taylor expansion is performed. Then grouping like polynomials, the unknown coefficients in the expansion are obtained, upon which the sum of these like polynomials is set to zero. The perturbation method is computationally more efficient than the direct Monte Carlo method. The disadvantages of this method are (1) it mainly focuses on the second-order estimate of the response and does not permit higher-order statistical estimates, and (2) the dependence on the random fluctuations being small.

M. Shinozuka, F. Yamazaki, G. Deodatis, et al. [1.20-1.21] studied the Neumann expansion method. They said that one advantage of this method is its simple formulation. They noted that in this method, the term $(K_0 + K_\theta)^{-1}$ is expanded using Neumann expansion to solve the stochastic algebraic equation $(K_0 + K_\theta)u = f$. The computational efficiency of the method depends on the range of random fluctuations. With this, the computational cost of Neumann expansion could even be more expensive than the direct Monte Carlo method for large-scale random fluctuations.

G. Deodatis and M. Shinozuka [1.38-1.39] studied frame structures and developed the weighted integral method for this purpose. In this method, the random material properties of beam elements are represented based on second-order statistical analysis and a

local averaging technique. In effect, the results are limited to second-order statistical quantities.

R.G. Ghanem, P.D. Spanos, *et al*. [1.33-1.37] subsequently formulated the polynomial chaos expansion method. The unknown stochastic field is expanded with multiple Hermite polynomials of random variables, i.e. polynomial chaos, and using Galerkin approach to solve the associated unknown coefficients. In addition, the application of the Karhunen-Loève expansion in the representation of random material properties is also developed.

C.C. Li and A.D. Kiureghian [1.40] presented another way of representing the random material properties called the least-squares approximation method. They reported that the error variance between the real stochastic field and the approximate one is the criterion of this method within each finite element. They added that this method is not as efficient as the Karhunen-Loève expansion method in terms of the number of random variables required.

D.B. Xiu and G.E. Karniadakis [1.44-1.46] generalized the polynomial chaos expansion method by replacing the Hermite polynomials with other orthogonal polynomials. They noted down that in this method, proof of well-posedness or convergence of the solution scheme is not present especially in the case of single random variable.

P.L. Liu and A.D. Kiureghian [1.26] employed the FORM and SORM to approximately calculate the probability integral with respect to the random solution. This is done to reduce the number of direct Monte Carlo simulations.

S. Valliappan, T.D. Pham, S.S. Rao, and J.P. Sawyer [1.41-1.43] applied the fuzzy set theory in the context of standard finite element analysis for engineering systems without precise or complete definitions. They called this application a fuzzy/interval finite element method. The results of this method are often vague, imprecise, qualitative and incomplete compared with the solution of a well defined engineering system because of the fact that the input information of this method is insufficient to properly define an engineering system.

Recently, Li [1.3] developed a novel solution strategy to the stochastic system of linear algebraic equations arising from stochastic finite element modeling called Jacobi-like joint diagonalization algorithm. In this method, the classical Jacobi algorithm for the computation of eigenvalue problem of a single matrix is modified to accommodate multiple real symmetric matrices.

1.3 The Aim and Layout of the Thesis

1.3.1 The Aim of the Thesis

The physical phenomena that involve random media are encountered in many areas of engineering as affirmed in Section 1.1, hence, a research concerning random medium mechanics is very important in engineering. Nevertheless, after the technical review in Section 1.2, it is apparent that existing knowledge and methods are not sufficient to analyze practical engineering structures composed of random media. Engineers customized the standard FEM into SFEM to suit for the particular problem of random media.

This thesis aims to study numerical techniques in stochastic finite element methods and develop applications in heat conduction and elastostatics problems. The task is naturally divided into three main parts: material modeling of random media; numerical solver of stochastic linear systems; and applications in heat conduction and elastostatics problems. In the first stage, the random material properties of heterogeneous materials are modeled as stochastic fields, for which an explicit representation is obtained by using the Fourier-Karhunen-Loève discretization scheme. In the second stage, the joint diagonalization solution strategy is investigated for the solution of the resulting stochastic linear systems, and along this direct attempts are made to improve the performance with joint tridiagonalization of multiple matrices. In the third stage, these numerical techniques are integrated into a SFEM framework, which is then tested in heat conduction and elastostatics problems.

1.3.2 Layout of the Thesis

This thesis is organized into six chapters as outline below.

Chapter 1: The background and motivation of the research are discussed which inspired the research undertaken. An incomplete but precise technical review on related research is also provided. Furthermore, the aim and outline of the thesis are included.

Chapter 2: This chapter deals with the methods on modeling the random media in stochastic fields. The mathematical model is discussed followed by the overview of existing schemes. The Fourier-Karhunen-Loève representation [1.3] is discussed and used to represent the stochastic fields of random media described by its random material properties.

Chapter 3: The stochastic finite element formulations for the steady-state heat conduction and linear elasticity are derived and discussed. The two formulations are then reduced to stochastic linear systems. Further, both the resulting stochastic linear systems have the same mathematical analogy, thus a generalized system is presented.

Chapter 4: In order to solve the stochastic system of linear algebraic equations derived in Chapter 3, the joint diagonalization strategy is explored in this chapter. It is shown that any multiple real symmetric matrices can be approximately transformed into diagonal and tri-diagonal matrix system using a sequence of orthogonal similarity transformations. A modified approach called joint tridiagonalization that simultaneously tridiagonalizes the multiple real symmetric matrices was attempted to develop in this chapter but due to time constraint the algorithm's convergence is still to be worked-out.

Chapter 5: Two numerical examples are presented using a SFEM framework where the joint diagonalization strategy is integrated to solve the resulting stochastic system of linear algebraic equations. In particular, the steady-state heat conduction and elastostatics problems of a concrete wastewater pipe are considered.

Chapter 6: Finally, this chapter discussed the summary and relevant conclusions of the thesis.

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Chapter 2

Representation of Stochastic Processes

In the framework of the finite element method wherein engineering systems composed of random media are analyzed, it is implicitly assumed that these random media have deterministic material parameters. These material parameters such as thermal conductivity, elastic modulus, etc. are assumed to have constant values over the structures domain. However, the material properties of the real structures composed of random media have several uncertainties. In view of these uncertainties, an appropriate random material model for the random media under consideration should be developed.

2.1 Mathematical Model

In random material modeling, there are at least three approaches used in the literature, namely, the fuzzy set theory [2.1-2.3], the theory of random matrices [2.4-2.5], and the probability theory [2.6-2.8]. Among these three approaches, the probability theory has been widely accepted in the SFEM community. Hence, a random material modeling approach of the random media based on probabilistic concepts with methods of mechanics is used in this thesis. The resulting mathematical model to describe the irregular variation of material properties through the random medium is defined on stochastic fields represented by random variables.

Let us assume that the Hilbert space of functions [2.9] defined over the domain $D \subset \mathbb{R}^n$, is denoted by **H**. Let $(\Omega, \mathcal{F}, P)$ denote a probability space, $\boldsymbol{x} \in D$ and $\theta \in \Omega$. Then, the space functions mapping $\Omega \to \mathbb{R}$ is denoted by $\boldsymbol{\Theta}$. Each map $\Omega \to \mathbb{R}$ defines a random variable.

The random material model involves a medium whose material properties demonstrates random spatial fluctuations and which may or may not be subjected to a random external forces. Expanding eqn. (1.1), the mathematical representation of this problem involves an operator equation which is not only a function over the medium domain $D \subset \mathbb{R}^n$, but also a function over a probability space $(\Omega, \mathcal{F}, P)$ expressed as

$$\boldsymbol{\Lambda}(\boldsymbol{x}, \theta) u(\boldsymbol{x}, \theta) = f(\boldsymbol{x}, \theta) \tag{2.1}$$

where $\boldsymbol{\Lambda}(\boldsymbol{x}, \theta)$ is a differential operator with coefficients exhibiting random fluctuations with respect to one or more independent variables defined on $\mathbf{H} \times \boldsymbol{\Theta}$. These coefficients are represented as random or stochastic processes (see Appendix for definition). The response $u(\boldsymbol{x}, \theta)$ is then solved as a function of both arguments. Likewise, the random coefficients of the differential operator are assumed to be of second order stochastic processes. An example of a random coefficient is the thermal conductivity tensor k in the steady-state heat conduction problem such that

$$k = k(\boldsymbol{x}, \theta) \qquad x \subset D, \ \theta \subset \Omega \tag{2.2}$$

Figure 2.1 shows an example of random material properties of a random medium, in which $k = k(\boldsymbol{x}, \theta_0)$ denotes a particular realization of the random thermal conductivity of a concrete slab.

Moreover, each of these coefficients $k_n(\boldsymbol{x}, \theta)$ can be defined by its first two lower order statistical moments, and decomposed into a deterministic and random components in the form

$$k_n(\boldsymbol{x}, \theta) = \bar{k}_n(\boldsymbol{x}) + \alpha_n(\boldsymbol{x}, \theta) \tag{2.3}$$

where $\bar{k}_n(\boldsymbol{x})$ denotes the mathematical expectation of the process $k_n(\boldsymbol{x}, \theta)$ and $\alpha_n(\boldsymbol{x}, \theta)$ is a zero-mean random process, having the same covariance function as the process $k_n(\boldsymbol{x}, \theta)$.

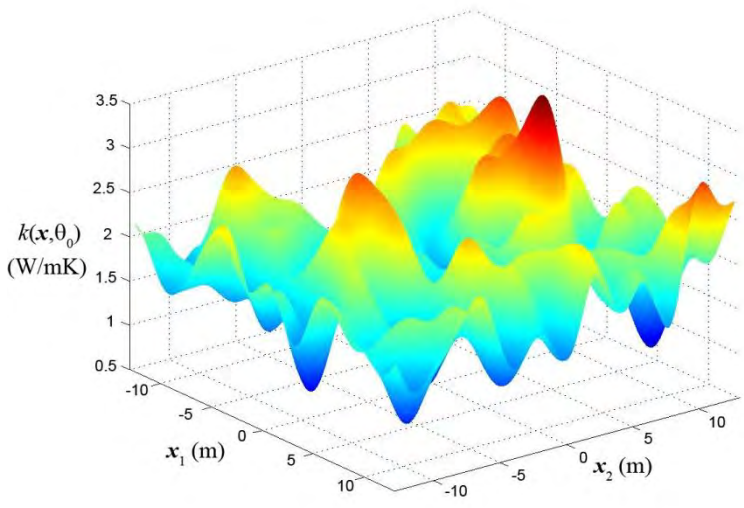

Figure 2.1 Stochastic thermal conductivity of a random medium.

In the random medium modeling, the continuity and differentiability of the stochastic fields are always taken into account for the reason that it will determine whether or not the SPDE holds. Further, in many cases the material property at any individual point $\boldsymbol{x}_0 \subset D$ of the stochastic field is described by a stationary random variable such as $k(\boldsymbol{x}_0, \theta)$ which is determined by its first two order statistical moments $k_0(\boldsymbol{x}_0)$ and $R(0)$. Furthermore, for simplicity it is assumed that the stochastic processes involved are of Gaussian distribution since a Gaussian field is completely defined by its mathematical expectation and covariance. Thus, the random medium assume as Gaussian field will make it possible to achieve explicit solutions in some typical random medium problems.

2.2 Representation of Stochastic Processes

In this thesis, the randomness of material properties through the random medium is modeled based on probability theory using stationary random variables in stochastic fields. The random medium model is defined by its mathematical expectation and covariance. In order to perform the associated differential operators in (2.1), the random medium property will be expanded according to an explicit representation of stochastic processes [2.10]. The thermal conductivity tensor k for example can be explicitly represented in an ideal form as

$$k(\boldsymbol{x},\theta) = l.i.m._{n\to+\infty} \sum_{i=0}^{n} \psi_i(\theta) k_i(\boldsymbol{x}) \tag{2.4}$$

where θ denotes the random dimension, $\psi_i(\theta)$ and $k_i(\boldsymbol{x})$ are a sequence of scalar-valued random variables and tensor-valued deterministic functions, respectively. These deterministic functions are often expressed in terms of piecewise polynomials. This series clearly separates the random and deterministic parts of $k(\boldsymbol{x},\theta)$ and thus makes the standard PDE tools applicable to an SPDE system.

This chapter attempts to demonstrate the existing methods of representation of stochastic processes of random medium of the form (2.4).

2.3 Overview of Existing Techniques

The major concern in dealing with the random medium problems is to realistically portray the irregular disparity of material properties through the medium, so that a suitable stochastic finite element formulation can be established for the problem under consideration.

Several stochastic field representation methods have been developed for the description of random material properties for the past years. The following paragraphs review the existing techniques and noting that majority of these methods is applicable for scalar stochastic fields.

M. Shinozuka [2.7] and F. Yamazaki, et.al. [2.11] used the middle point method to discretize the definition domain of the stochastic field with finite element (FE) mesh, and the stochastic field is simply approximated by the random variable at the element center in each element. Then followed by Cholesky decomposition of covariance matrix to determine the central random variables, which is constructed by directly sampling the given covariance function.

The local averaging method has been successfully applied by E. Vanmarcke and M. Grigoriu [2.6] in simple beam elements. It follows the same discretization procedure as the

middle point method in which random variables represent the stochastic field in each element, which is calculated within the element from a spatial average of the local stochastic field.

In addition, the shape function method [2.12-2.14] adopts the same discretization procedure as the first two methods mentioned above. Although, the nodal random variables and the associated shape functions are used to interpolate the stochastic field in each element. The exact value from the given covariance is then taken from the covariance between each pair of nodal random variables.

The least-squares discretization method [2.15] is an efficient discretization of random fields and more practical than a series expansion method employing the Karhunen-Loève theorem. Following the discretization of the definition domain of the stochastic field with FE mesh, the nodal random variables are determined by means of an optimization procedure. The error variance between the interpolated stochastic field and the exact stochastic field is then used to measure the approximation accuracy within each element.

The trigonometric series approximation method [2.16-2.19] is unique compared to the methods mentioned above in the sense that no FE mesh is required in this method. This method used the general trigonometric series with random coefficients to approximate the stochastic field. This is achieved through direct discretization of the spectral representation of the wide-sense stationary stochastic field.

Ghanem and Spanos [2.20] are the first to introduce into the SFEM research the Karhunen-Loève expansion method [2.20-2.25] which is based on Karhunen-Loève (K-L) expansions [2.26] of second-order stochastic fields. Since then it has been widely used to describe random material properties [2.21-2.25]. It will be explained shortly below.

Suppose that the random medium D where $D \subset \mathbb{R}^n$ has the random material property such as the thermal conductivity tensor represented by a second-order stochastic field $k(\boldsymbol{x}, \theta)$ where $\boldsymbol{x} \in D$ and $\theta \in \Omega$. Following eqn. (2.3), the stochastic field $k(\boldsymbol{x}, \theta)$ can be expanded by its mean $E\big(k(\boldsymbol{x}, \theta)\big)$ and the stochastic process $\alpha(\boldsymbol{x}, \theta)$ with zero mean and

same covariance $Cov\big(k(\boldsymbol{x}_1,\theta),k(\boldsymbol{x}_2,\theta)\big)$ of the process $k(\boldsymbol{x},\theta)$. Hence, the process $k(\boldsymbol{x},\theta)$ can be expressed as

$$k(\boldsymbol{x},\theta) = E\big(k(\boldsymbol{x},\theta)\big) + \sum_{i=1}^{+\infty}\sqrt{\lambda_i}\boldsymbol{\phi}_i(\boldsymbol{x})\boldsymbol{\xi}_i(\theta) \tag{2.5}$$

where $\boldsymbol{\xi}_i(\theta)$ is the set of uncorrelated random variables, $\boldsymbol{\phi}_i(\boldsymbol{x})$ and λ_i are the eigenfunctions and eigenvalues, respectively of the characteristic equation

$$\int_D Cov\big(k(\boldsymbol{x}_1,\theta),k(\boldsymbol{x}_2,\theta)\big)\boldsymbol{\phi_i}(\boldsymbol{x}_1)d\boldsymbol{x}_1 = \lambda_i\boldsymbol{\phi_i}(\boldsymbol{x_2}) \tag{2.6}$$

such that $\lambda_i > 0$.

It is emphasized that the mean and covariance of $k(\boldsymbol{x},\theta)$ are inadequate to fully define a general second-order stochastic field. Most often, additional information is needed in order to solve the problem. If $k(\boldsymbol{x},\theta)$ is a Gaussian field, then $\boldsymbol{\xi}_i(\theta)$ are Gaussian random variables [2.26]. In this way, the probability distribution of $\boldsymbol{\xi}_i(\theta)$ can be easily determined.

In practice, the summation term in (2.5) is truncated at a finite number *n*. The decay of the eigenvalues from eqn. (2.6) is used to determine the number of terms *n* to ensure the truncation error is acceptably small.

The Fourier-Karhunen-Loève (F-K-L) representation scheme developed by Li [2.10] is probably the best method for representing random processes at present. It is completely mesh-free, accurate, and computationally more efficient compare to K-L expansion method. This method is used to represent stochastic processes in Chapter 3 and will be explained in Section 2.4.

In the technical review above, it is noted that the discretization format using FE meshes have been widely used in the solution of this stochastic field representation problem. Particularly, the same mesh employed in solving the SPDE system can be used to describe the random material properties, which in turn makes it easier to deal with random media in a similar framework as the standard finite element method.

2.4 Fourier-Karhunen-Loève Representation of Random Media

The F-K-L representation of random media will be established in this section for a random material property with scalar stochastic field. It is also possible to extend this representation of random material properties for several parameters that can be described by a generalized tensor stochastic field, but this will not be covered in this section.

Let an infinite *n*-dimensional random media with random material property be described by a second-order stochastic field $k(\boldsymbol{x},\theta)$ where $\boldsymbol{x} \in D$ and $\theta \in \Omega$. Denoting that the expectation function and covariance function of $k(\boldsymbol{x},\theta)$ are $k_0(\boldsymbol{x})$ and $R(\boldsymbol{\tau}) = R(\boldsymbol{x}_1 - \boldsymbol{x}_2)$, respectively. Then, the *F-K-L representation of random medium in* $\mathbb{R}^n$ can be expressed as

$$k(\boldsymbol{x},\theta) = k_0(\boldsymbol{x}) + \int_{\mathbb{R}^n} e^{\sqrt{-1}x\cdot y} d\boldsymbol{Z}(\boldsymbol{y},\theta) \tag{2.7}$$

Also, the covariance function can be written as

$$R(\boldsymbol{\tau}) = \int_{\mathbb{R}^n} e^{\sqrt{-1}\tau\cdot y} dF(\boldsymbol{y}) = \int_{\mathbb{R}^n} f(\boldsymbol{y}) e^{\sqrt{-1}\tau\cdot y} d\boldsymbol{y} \tag{2.8}$$

The terms $k(\boldsymbol{x},\theta)$ and $R(\boldsymbol{\tau})$ are in the form of the inverse Fourier transform and expressed in the frequency space. In addition, the spectral distribution function of $k(\boldsymbol{x},\theta)$ is denoted by $F(\boldsymbol{y})$. Furthermore, the spectral density function $f(\boldsymbol{y})$ can be easily derived via the Fourier transform

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} R(\boldsymbol{\tau}) e^{\sqrt{-1}\tau\cdot y} d\,\boldsymbol{\tau} \tag{2.9}$$

Interestingly, eqns. (2.7-2.8) provide an explicit solution for the K-L expansion of $k(\boldsymbol{x},\theta)$ where $\boldsymbol{x} \in \mathbb{R}^n$ in terms of Fourier integrals without solving eqn. (2.6). Noting also that the terms $e^{\sqrt{-1}x\cdot y}$ and $d\boldsymbol{Z}(\boldsymbol{y},\theta)$ in eqn. (2.7) correspond to $\boldsymbol{\phi}_i(\boldsymbol{x})$ and $\sqrt{\boldsymbol{\lambda}_i}\boldsymbol{\xi}_i(\theta)$ in eqn. (2.5), respectively.

Note that for a typical random media encountered in practice, the spectral density function $f(\boldsymbol{y})$ in eqn. (2.9) can be exactly obtained since the Fourier transforms of many

typical functions can be calculated analytically. Otherwise, calculate $f(\boldsymbol{y})$ numerically using the standard Fast Fourier Transform (FFT).

The inverse Fourier transform (2.7) of $k(\boldsymbol{x},\theta)$ can be deduced to a convergent Fourier series of $k(\boldsymbol{x},\theta)$. Due to the orthogonality of the Fourier basis and using the Principal Component Analysis (PCA), the *F-K-L representation of random media in n-*intervals is expressed as

$$k_N(\boldsymbol{x},\theta) = k_0(\boldsymbol{x}) + \sum_{i=1}^{N}\sqrt{\lambda_i}\,\boldsymbol{\phi}_i(\boldsymbol{x})\boldsymbol{\xi}_i(\theta) \tag{2.10}$$

$$\approx k_0(\boldsymbol{x}) + \sum_{i=1}^{N^*}\sqrt{\lambda_i}\,\boldsymbol{\phi}_i(\boldsymbol{x})\boldsymbol{\xi}_i(\theta) = k_{N^*}(\boldsymbol{x},\theta)$$

where N^* is the smallest integer such that

$$\frac{\sum_{i=1}^{N^*}\lambda_i}{\sum_{i=1}^{N}\lambda_i} \geq \mu^* \qquad 0 < \mu^* \leq 1 \tag{2.11}$$

where the ratio μ^* is the needed accuracy for approximating the total variance of $k_N(\boldsymbol{x},\theta)$. Also,

$$\frac{\int_{\mathbf{F}\boldsymbol{n}} f(\boldsymbol{y})d\boldsymbol{y}}{\int_{\mathbb{R}\boldsymbol{n}} f(\boldsymbol{y})d\boldsymbol{y}} = \frac{\int_{\mathbf{F}\boldsymbol{n}} f(\boldsymbol{y})d\boldsymbol{y}}{R(0)} \geq \mu \qquad 0 < \mu < 1 \tag{2.12}$$

where constant μ is the required accuracy for approximating the spectrum identified by $f(\boldsymbol{y})$.

The term $k_{N^*}(\boldsymbol{x},\theta)$ provides a series solution for the K-L expansion of $k(\boldsymbol{x},\theta)$, this is due to the completeness and orthogonality of the Fourier basis and the PCA procedure. The terms $\boldsymbol{\xi}_i(\theta)$ in eqn. (2.10) are stochastically independent standard Gaussian random variables. This is a direct consequence of $k(\boldsymbol{x},\theta)$ being a Gaussian field in the random media model. The truncation error $k(\boldsymbol{x},\theta) - k_{N^*}(\boldsymbol{x},\theta)$ is explicitly controlled by μ and μ^*. Particularly, the ratio μ controls the error of the spectrum (i.e. eigenvalues and eigenfunctions) of $k(\boldsymbol{x},\theta)$ and the ratio μ^* controls the error of the total variance of $k(\boldsymbol{x},\theta)$.

The case for an irregular domain D, the F-K-L representation above constructed via the n-interval is not equivalent to performing an exact K-L expansion of $k(\boldsymbol{x},\theta)$ where

$\boldsymbol{x} \in D$. Therefore, depending on the specific domain D, the F-K-L representation may be less economical than the exact K-L expansion in terms of the number of random variables included in the series expression.

2.5 Summary

In this chapter, an explicit stochastic field F-K-L representation scheme is presented to describe the random material properties of the random media model. This scheme is completely mesh-free and also independent of the detailed shape of the random structure under consideration compare to various FE-mesh based stochastic field representation schemes. Also, for the case of only one random material parameter involved in the random media model, the F-K-L scheme provides a semi-analytical solution of the K-L expansion of the corresponding stochastic field. In contrast, the F-K-L representation scheme is not only more accurate but is also computationally more efficient than the widely used K-L expansion method which is based on FE meshes. This is due to the harmonic essence of wide-sense stationary stochastic fields [2.10].

The F-K-L representation is attained with a prior error control. The error of the spectrum of stochastic fields and the error of the total variance are explicitly controlled by two parameters μ^* and μ.

It is worth mentioning that, it is also possible to have an F-K-L representation for the general elastic tensor of random media to make the random media model applicable to any elastic constitutive relation.

Appendix: Definitions

In order to easily understand the mathematical terms used in this chapter a brief definition is given below.

Definition 2.1: Consider a probability space $(\Omega, \mathcal{F}, P)$ is given. By definition, a random variable is a function X from Ω to the real line $\mathbb{R}$ that is $\mathcal{F}$ measurable, meaning that for any number a,

$$\{\omega : X(\omega) \leq a\} \in \mathcal{F}. \tag{2.13}$$

If Ω is finite or countably infinite, then $\mathcal{F}$ can be the set of all subsets of Ω, in which case any real-valued function on Ω is a random variable [2.29].

Definition 2.2: The expectation (also called expected value, average value, mean value or simply mean) of a random variable X if there is a finite set $\{x_1,...,x_m\}$ such that $X(\omega) \in \{x_1,...,x_m\}$ can be expressed mathematically as

$$E(X) = \bar{X} = \sum_x xP\{X = x\} \tag{2.14}$$

Definition 2.3: Let X and Y be random variables on the same probability space, then the covariance of X and Y is defined as

$$Cov(X,Y) = E\left(\left(X - E(X)\right)\left(Y - E(Y)\right)\right) \tag{2.15}$$

Definition 2.4: The stochastic process, or sometimes random process X is an indexed collection $X = (X_t : t \in \mathbb{R})$ of random variables, all on the same probability space $(\Omega, \mathcal{F}, P)$ [2.29]. Stochastic field is simply a stochastic process and can be defined in an n-dimensional ($n \in \mathbb{N}$) domain.

Definition 2.5: Let the material property, for example thermal conductivity of the random medium D where $D \subset \mathbb{R}^n$ be represented by a second-order stochastic field $k(\boldsymbol{x}, \theta)$ where $\boldsymbol{x} \in D$ and $\theta \in \Omega$. Note that the stochastic field $k(\boldsymbol{x}, \theta)$ is a function over the random medium domain $D \subset \mathbb{R}^n$ and probability space $(\Omega, \mathcal{F}, P)$, respectively. The stochastic field

is governed by several conditions [2.10]. Firstly, the expectation function of $k(\boldsymbol{x},\theta)$ is up to m-th order continuous derivatives in D, such that

$$E\big(k(\boldsymbol{x},\theta)\big) = k_0(\boldsymbol{x}) \tag{2.16}$$

where $k_0(\boldsymbol{x}) \in C^m(D)$. Secondly, the covariance function of $k(\boldsymbol{x},\theta)$ is defined as

$$Cov\big(k(\boldsymbol{x}_1,\theta), k(\boldsymbol{x}_2,\theta)\big) \equiv R(\boldsymbol{\tau}) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in D \tag{2.17}$$

where $\boldsymbol{\tau} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. Thirdly, the covariance function $R(\boldsymbol{\tau})$ is C^{2m} continuous at $\boldsymbol{\tau} = 0$, such that, it is up to $2m$-th order continuous derivatives at the origin. Lastly, for simplicity the stochastic field $k(\boldsymbol{x},\theta)$ is assumed to be a Gaussian field.

Definition 2.6: A random process, $X = (X_t : t \in \mathbb{R})$ is stationary if all its n-order marginals do not depend on a translation by a constant, such that for any $n > 0$ and time indices $t_1, \cdots, t_n$,

$$f_{x_{t_1},\cdots,x_{t_n}}(x_1,\cdots,x_n) = f_{x_{t_1+\tau},\cdots,x_{t_n+\tau}}(x_1,\cdots,x_n) \qquad \forall \tau \in \mathbb{R}. \tag{2.18}$$

Or simply, the stochastic process is said to be stationary process if the joint probability distribution does not change when shifted in space or time. Thus, the parameters such as the mean and variance do not change over position or time.

Definition 2.7: A random process, $X = (X_t : t \in \mathbb{R})$ is wide sense stationary (WSS) if its mean function $E\big(X(t)\big)$ is constant and its autocorrelation $R_X(t,s)$ is a function of $|s-t|$, i.e. $R_X(\tau) = R_X(t, t+\tau)$.

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Chapter 3
Stochastic Finite Element Formulation

The FEM is widely known and very effective in solving virtually every engineering field problems. However, the conventional FEM cannot be used for solving problems with dominant random parameters. In order to accommodate this type of problems, the deterministic standard FEM is generalized as the SFEM by integrating the random variations in materials and geometric properties of the model and the random excitation that may act on it. In general, the major steps involved in the SFEM [3.1] formulation are outlined as follows:

1) Determination of suitable probabilistic model for the random fields;
2) Discretization of random fields;
3) Formulation of the FE equations of motion and its solution;
4) Estimation of system response probabilistic characteristics; and
5) Prediction using the results, such as structural reliability.

Following the representation of random media using F-K-L representation scheme in Chapter 2, this chapter covers step 3) above, i.e. the formulation of the FE equations of motion and its solution. Introducing the probability space in the governing equations of any physical phenomena, the equivalent SPDE system of any random media can be derived. The SPDE system is then discretized to achieve a stochastic system of linear algebraic

equations using the standard finite element. The resulting stochastic system of linear algebraic equations will then be solved easily using the solver in Chapter 4.

3.1 Stochastic Partial Differential Equation System

Two typical problems encountered in engineering applications will be presented in this chapter, namely (1) elastostatics and (2) steady-state heat conduction problems. These types of problems share the same mathematical analogy. Hence, at the end a generalized stochastic system of linear algebraic equations is applicable to both problems.

3.1.1 Elastostatics

Mathematical models of physical systems are typically governed by various PDE systems, so with the engineering structures composed of random media. In the analysis of an engineering structures based on deterministic mathematical models, input data such as the material properties are usually given which provide unique structural responses for a particular boundary conditions. For example, it is well known in continuum mechanics that the governing equations and required boundary conditions for elastostatics are mathematically expressed as

$$\begin{cases} \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{0} \\ \boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x}):\boldsymbol{\varepsilon}(\boldsymbol{x}) \\ \boldsymbol{\varepsilon}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{u}(\boldsymbol{x})\boldsymbol{\nabla} + \boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x})\right) \end{cases} \qquad \begin{cases} \boldsymbol{u}(\boldsymbol{x}) - \tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0} \ \ on\ \partial D_u \\ \boldsymbol{\sigma}(\boldsymbol{x}) - \tilde{\boldsymbol{\sigma}}(\boldsymbol{x}) = \boldsymbol{0} \ \ on\ \partial D_\sigma \end{cases} \tag{3.1}$$

for $\boldsymbol{x} \in D \subset \mathbb{R}^n$

in which $\boldsymbol{u}(\boldsymbol{x})$ and $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ represent the displacement vectors, $\boldsymbol{\varepsilon}(\boldsymbol{x})$ is the strain tensor, $\boldsymbol{\sigma}(\boldsymbol{x})$ and $\tilde{\boldsymbol{\sigma}}(\boldsymbol{x})$ are the stress tensors, $\boldsymbol{C}(\boldsymbol{x})$ is the elastic tensor, and $\boldsymbol{b}(\boldsymbol{x})$ is the load vector defined as $\boldsymbol{b}(\boldsymbol{x}) = \rho(\boldsymbol{x})\boldsymbol{g}$ where $\rho(\boldsymbol{x})$ and $\boldsymbol{g}$ are the density tensor and acceleration due to gravity vector, respectively; $D \subset \mathbb{R}^n$ is the material domain, ∂D_u is the displacement boundary of D and ∂D_σ is the stress boundary of D.

Ideal situations are rarely encountered in practice; hence, the need to address uncertainties is clearly recognized. The modeling approach of engineering structures

composed of random media is based on a combination of probabilistic concepts with the methods of mechanics. Most random material models are based on stochastic fields to describe the irregular variation of material properties. In order to integrate this stochastic fields, the specified vectors and tensors will be expressed as a function over the medium domain $D \subset \mathbb{R}^n$ and probability space $(\Omega, \mathcal{F}, P)$, respectively, such that:

$$\begin{cases} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{x}, \theta) \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}, \theta) \\ \rho = \rho(\boldsymbol{x}, \theta) \\ \boldsymbol{C} = \boldsymbol{C}(\boldsymbol{x}, \theta) \end{cases} \quad \text{where } \boldsymbol{x} \in D \subset \mathbb{R}^n, \theta \in \Omega \subset \mathbb{R}^n \tag{3.2}$$

Accordingly, for random medium expression (3.1) is transformed to a SPDE system as follows:

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{x}, \theta) + \rho(\boldsymbol{x}, \theta)\boldsymbol{g} = \boldsymbol{0} \\ \boldsymbol{\sigma}(\boldsymbol{x}, \theta) = \boldsymbol{C}(\boldsymbol{x}, \theta) : \boldsymbol{\varepsilon}(\boldsymbol{x}, \theta) \\ \boldsymbol{\varepsilon}(\boldsymbol{x}, \theta) = \frac{1}{2}\left(\boldsymbol{u}(\boldsymbol{x}, \theta)\boldsymbol{\nabla} + \boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x}, \theta)\right) \end{cases} \quad \begin{cases} \boldsymbol{u}(\boldsymbol{x}, \theta) - \tilde{\boldsymbol{u}}(\boldsymbol{x}, \theta) = \boldsymbol{0} \ \ on\ \partial D_u \\ \boldsymbol{\sigma}(\boldsymbol{x}, \theta) - \tilde{\boldsymbol{\sigma}}(\boldsymbol{x}, \theta) = \boldsymbol{0} \ \ on\ \partial D_\sigma \end{cases} \tag{3.3}$$

for $\boldsymbol{x} \in D \subset \mathbb{R}^n, \theta \in \Omega \subset \mathbb{R}^n$

which is now expressed in terms of stochastic fields. Note that these mathematical quantities have the same physical meanings as in eqn. (3.1). The randomness in the above SPDEs is introduced by the spatially irregular variation of material properties, aside from the possible random boundary conditions $\tilde{\boldsymbol{u}}(\boldsymbol{x}, \theta)$ and $\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}, \theta)$, which can be treated separately. Hence, the probabilistic properties of the random solutions $\boldsymbol{u}(\boldsymbol{x}, \theta)$, $\boldsymbol{\varepsilon}(\boldsymbol{x}, \theta)$ and $\boldsymbol{\sigma}(\boldsymbol{x}, \theta)$ can be determined from the probability distributions of the stochastic fields $\rho(\boldsymbol{x}, \theta)$ and $\boldsymbol{C}(\boldsymbol{x}, \theta)$.

3.1.2 Steady-State Heat Conduction

The governing equations and required boundary conditions for steady-state heat conduction are expressed as

$$\begin{cases} -\nabla \cdot k(\boldsymbol{x})\nabla \boldsymbol{T}(\boldsymbol{x}) + \boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{0} & \\ \boldsymbol{T}(\boldsymbol{x}) - \tilde{\boldsymbol{T}}(\boldsymbol{x}) = \boldsymbol{0} & \text{on } \partial D_T \\ \boldsymbol{q}_n(\boldsymbol{x}) - \tilde{\boldsymbol{q}}(\boldsymbol{x}) = \boldsymbol{0} & \text{on } \partial D_q \end{cases} \tag{3.4}$$

$\boldsymbol{x} \in D \subset \mathbb{R}^n$

in which $\boldsymbol{T}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{T}}(\boldsymbol{x})$ represent the temperature vectors, $k(\boldsymbol{x})$ is the thermal conductivity tensor, $\boldsymbol{q}_n(\boldsymbol{x})$ and $\widetilde{\boldsymbol{q}}(\boldsymbol{x})$ are the heat flux vectors, and $\boldsymbol{Q}(\boldsymbol{x})$ is the heat source vector; $D \subset \mathbb{R}^n$ is the material domain, ∂D_T is the temperature and ∂D_q is the heat flux boundaries of D, respectively.

The random variation of material properties can be described by integrating the stochastic fields into the thermal conductivity tensor, temperature vector and heat source vector which are expressed as a function over the medium domain $D \subset \mathbb{R}^n$ and a function over a probability space $(\Omega, \mathcal{F}, P)$ written as:

$$\begin{cases} k = k(\boldsymbol{x}, \theta) \\ \boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, \theta) \\ \boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{x}, \theta) \end{cases} \quad \text{where} \quad \boldsymbol{x} \in D \subset \mathbb{R}^n, \theta \in \Omega \subset \mathbb{R}^n \qquad (3.5)$$

As an example of this, the material properties $k = k(\boldsymbol{x}, \theta_0)$ as shown in Figure 2.1 represent a particular realization of random thermal conductivity of a concrete slab.

Consequently, for a spatially varying medium the corresponding SPDE system of the governing equations and associated boundary conditions of steady-state heat conduction is written as follows:

$$\begin{cases} -\nabla \cdot k(\boldsymbol{x}, \theta) \nabla \boldsymbol{T}(\boldsymbol{x}, \theta) + \boldsymbol{Q}(\boldsymbol{x}, \theta) = \boldsymbol{0} & \\ \boldsymbol{T}(\boldsymbol{x}, \theta) - \widetilde{\boldsymbol{T}}(\boldsymbol{x}, \theta) = \boldsymbol{0} & \text{on } \partial D_T \\ \boldsymbol{q}_n(\boldsymbol{x}, \theta) - \widetilde{\boldsymbol{q}}(\boldsymbol{x}, \theta) = \boldsymbol{0} & \text{on } \partial D_q \end{cases} \qquad (3.6)$$

$$\text{for } \boldsymbol{x} \in D \subset \mathbb{R}^n, \ \theta \in \Omega \subset \mathbb{R}^n$$

It is observed that eqns. (3.3) and (3.6) are analogous mathematically, hence, a generalized formulation is necessary to solve the two different physical problems.

As depicted above, the domain of a random medium can be described also as stochastic fields. In SFEM, the random process is represented by a countable set of random variables; thus, discretizing the process.

3.2 F-K-L Representation

The random material properties $\rho(\boldsymbol{x}, \theta)$, $\boldsymbol{C}(\boldsymbol{x}, \theta)$ and $k(\boldsymbol{x}, \theta)$ of both the elastostatics and steady-state heat conduction problems are typically defined by their first-

and second- order statistical moments. Accordingly, the density vector $\rho(\boldsymbol{x},\theta)$, elastic tensor $\boldsymbol{C}(\boldsymbol{x},\theta)$ and thermal conductivity tensor $k(\boldsymbol{x},\theta)$ are defined as follows:

$$E\big(\rho(\boldsymbol{x},\theta)\big) = \rho_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in D \tag{3.7}$$

$$Cov\big(\rho(\boldsymbol{x}_1,\theta),\rho(\boldsymbol{x}_2,\theta)\big) = R_\rho(\boldsymbol{\tau}) \qquad \forall \boldsymbol{x}_1,\boldsymbol{x}_2 \in D; \tag{3.8}$$

$$E\big(\boldsymbol{C}(\boldsymbol{x},\theta)\big) = \boldsymbol{C}_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in D \tag{3.9}$$

$$Cov\big(\boldsymbol{C}(\boldsymbol{x}_1,\theta),\boldsymbol{C}(\boldsymbol{x}_2,\theta)\big) = R_C(\boldsymbol{\tau}) \qquad \forall \boldsymbol{x}_1,\boldsymbol{x}_2 \in D; \tag{3.10}$$

$$E\big(k(\boldsymbol{x},\theta)\big) = k_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in D \tag{3.11}$$

$$Cov\big(k(\boldsymbol{x}_1,\theta),k(\boldsymbol{x}_2,\theta)\big) = R_k(\boldsymbol{\tau}) \qquad \forall \boldsymbol{x}_1,\boldsymbol{x}_2 \in D \tag{3.12}$$

where $\boldsymbol{\tau} = \boldsymbol{x}_1 - \boldsymbol{x}_2$; in which the scalar entries of vector $\rho_0(\boldsymbol{x})$, tensor $\boldsymbol{C}_0(\boldsymbol{x})$ and tensor $k_0(\boldsymbol{x})$ are the expectation functions of the corresponding scalar entries in $\rho(\boldsymbol{x},\theta)$, $\boldsymbol{C}(\boldsymbol{x},\theta)$ and $k(\boldsymbol{x},\theta)$, respectively. Also, the elements of matrices $R_\rho(\boldsymbol{\tau})$, $R_C(\boldsymbol{\tau})$ and $R_k(\boldsymbol{\tau})$ are the covariance functions of the corresponding scalar entries in $\rho(\boldsymbol{x},\theta)$, $\boldsymbol{C}(\boldsymbol{x},\theta)$ and $k(\boldsymbol{x},\theta)$, respectively.

According to eqns. (3.7-3.12) and the F-K-L representation scheme of random media, the density vector $\rho(\boldsymbol{x},\theta)$, elastic tensor $\boldsymbol{C}(\boldsymbol{x},\theta)$ and thermal conductivity tensor $k(\boldsymbol{x},\theta)$ can be expressed as follows:

$$\rho(\boldsymbol{x},\theta) \approx \rho_0(\boldsymbol{x}) + \sum_{i=1}^{N_\rho^*} \sqrt{\lambda_i^\rho}\, \xi_i^\rho(\theta)\rho_i(\boldsymbol{x}) \tag{3.13}$$

$$\boldsymbol{C}(\boldsymbol{x},\theta) \approx \boldsymbol{C}_0(\boldsymbol{x}) + \sum_{i=1}^{N_C^*} \sqrt{\lambda_i^C}\, \boldsymbol{\xi}_i^C(\theta)\boldsymbol{C}_i(\boldsymbol{x}) \tag{3.14}$$

$$k(\boldsymbol{x},\theta) \approx k_0(\boldsymbol{x}) + \sum_{i=1}^{N_k^*} \sqrt{\lambda_i^k}\, \xi_i^k(\theta) k_i(\boldsymbol{x}) \tag{3.15}$$

in which λ_i^ρ, λ_i^C and λ_i^k are all positive constants; $\xi_i^\rho(\theta)$, $\boldsymbol{\xi}_i^C(\theta)$ and $\xi_i^k(\theta)$ are all assumed to be stochastically independent Gaussian random variables; $\rho_i(\boldsymbol{x})$, $\boldsymbol{C}_i(\boldsymbol{x})$ and $k_i(\boldsymbol{x})$ are all series of orthonormal deterministic functions; and the integers N_ρ^*, N_C^* and N_k^* are determined explicitly by the error-control parameters in the F-K-L representation scheme.

The boundary conditions of eqns. (3.3) and (3.6) are allowed to be random, but this possible randomness can be treated separately as it has no direct effect on the associated

differential operators. Consequently, the definition method of the stochastic fields $\tilde{\boldsymbol{u}}(\boldsymbol{x},\theta)$, $\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},\theta)$, $\tilde{\boldsymbol{T}}(\boldsymbol{x},\theta)$, and $\tilde{\boldsymbol{q}}(\boldsymbol{x},\theta)$ are not specified in the random media model. However, other appropriate methods can be used to define it, aside from its statistical moments.

3.3 Stochastic System of Linear Algebraic Equations

Equations (3.3, 3.6-3.15) formed the complete SPDE systems for elastostatics and steady-state heat conduction of random media. The generalized finite element discretization for the PDE system can be used to discretize these SPDE systems, this is due to the fact that SPDE system is a generalization of the conventional PDE system. The details of the well-established FEM can be found in various FE textbooks ([3.2-3.5] for instance). These two SPDE systems will be derived separately but without showing much of the details and later on a generalization of the formulation will be considered.

3.3.1 Elastostatics

Following the standard FEM, the random medium D is discretized first with a finite element mesh. Employing variational formulation of the FEM to eqn. (3.3), the functional of the total potential of the random media can be written as

$$\prod_p = \sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \frac{1}{2}\left(\int_{D^e} \left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{D}(\boldsymbol{x},\theta)\boldsymbol{B}(\boldsymbol{x})d(vol)\right)\boldsymbol{u}^e(\theta) \quad (3.16)$$

$$-\sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \int_{D^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{g}\,\rho(\boldsymbol{x},\theta)d(vol)$$

$$-\sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \int_{\partial D_\sigma^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \tilde{\boldsymbol{\sigma}}(\boldsymbol{x},\theta)d(area)$$

in which Σ_e represents the summation over all the elements, $\boldsymbol{B}(\boldsymbol{x})$ is the strain matrix, $\boldsymbol{D}(\boldsymbol{x},\theta)$ is the elastic matrix, $\boldsymbol{u}^e(\theta)$ is the nodal displacement vector of each element

and $\boldsymbol{N}(\boldsymbol{x})$ is the shape function matrix. The elastic matrix based on eqn. (3.14) can be expressed as

$$\boldsymbol{D}(\boldsymbol{x},\theta) \approx \boldsymbol{D}_0(\boldsymbol{x}) + \sum_{i=1}^{N_C^*} \sqrt{\lambda_i^C}\, \xi_i^C(\theta)\boldsymbol{D}_i(\boldsymbol{x}) \tag{3.17}$$

in which the elastic matrices $\boldsymbol{D}_0(\boldsymbol{x})$ and $\boldsymbol{D}_i(\boldsymbol{x})$ are derived respectively from the elastic tensors $\boldsymbol{C}_0(\boldsymbol{x})$ and $\boldsymbol{C}_i(\boldsymbol{x})$. By employing eqns. (3.13) and (3.17) into (3.16), the total potential is now expressed as

$$\prod_p = \sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \frac{1}{2}\left(\boldsymbol{K}_0^e + \sum_{i=1}^{N_C^*} \xi_i^C(\theta)\, \boldsymbol{K}_i^e\right) \boldsymbol{u}^e(\theta) \tag{3.18}$$

$$- \sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \left(\boldsymbol{F}_0^e + \sum_i^{N_\rho^*} \xi_{i=1}^{\rho}(\theta)\, \boldsymbol{F}_i^e\right) - \sum_e \left(\boldsymbol{u}^e(\theta)\right)^{\mathrm{T}} \tilde{\boldsymbol{F}}^e(\theta)$$

in which the elemental stiffness matrices $\boldsymbol{K}_0^e$ and $\boldsymbol{K}_i^e$, the elemental volume-force vectors $\boldsymbol{F}_0^e$ and $\boldsymbol{F}_i^e$, and the elemental external-load vector $\tilde{\boldsymbol{F}}^e(\theta)$ are defined as follows:

$$\boldsymbol{K}_0^e = \int_{D^e} \left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{D}_0(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}) d(vol) \tag{3.19}$$

$$\boldsymbol{K}_i^e = \sqrt{\lambda_i^C} \int_{D^e} \left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{D}_i(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\, d(vol) \qquad i = 1,2,\cdots,N_C^* \tag{3.20}$$

$$\boldsymbol{F}_0^e = \int_{D^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{g}\, \rho_0(\boldsymbol{x}) d(vol) \tag{3.21}$$

$$\boldsymbol{F}_i^e = \sqrt{\lambda_i^\rho} \int_{D^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{g}\, \rho_i(\boldsymbol{x}) d(vol) \qquad i = 1,2,\cdots,N_\rho^* \tag{3.22}$$

$$\tilde{\boldsymbol{F}}^e(\theta) = \int_{\partial D_\sigma^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \tilde{\boldsymbol{\sigma}}(\boldsymbol{x},\theta) d(area) \tag{3.23}$$

Minimizing the total potential $\prod_p$ by differentiating with respect to the displacement on both sides of Eq. (3.18) and equating to zero, then the following stochastic system of linear algebraic equations hold

$$\left(\boldsymbol{K}_0 + \sum_{i=1}^{N_C^*} \xi_i^C(\theta)\, \boldsymbol{K}_i\right) \boldsymbol{U}(\theta) = \left(\boldsymbol{F}_0 + \sum_{i=1}^{N_\rho^*} \xi_i^\rho(\theta)\, \boldsymbol{F}_i\right) + \tilde{\boldsymbol{F}}(\theta) \tag{3.24}$$

in which the global stiffness matrices $\boldsymbol{K}_0$ and $\boldsymbol{K}_i$ are assembled respectively from elemental $\boldsymbol{K}_0^e$ and $\boldsymbol{K}_i^e$; the global volume-force vectors $\boldsymbol{F}_0$ and $\boldsymbol{F}_i$ are assembled respectively from elemental $\boldsymbol{F}_0^e$ and $\boldsymbol{F}_i^e$; the global external-load vector $\widetilde{\boldsymbol{F}}(\theta)$ is assembled from elemental $\widetilde{\boldsymbol{F}}^e(\theta)$; and $\boldsymbol{U}(\theta)$ is the unknown nodal displacement vector.

3.3.2 Steady-State Heat Conduction

Similarly, the random medium D is discretized first with a finite element mesh. Then, employing variational formulation of the FEM to eqn. (3.6), the functional of the total potential of the random media can be expressed as

$$\prod_p = \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \frac{1}{2} \left(\int_{D^e} \left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} k(\boldsymbol{x},\theta) \boldsymbol{B}(\boldsymbol{x}) d(vol) \right) \boldsymbol{T}^e(\theta) \qquad (3.25)$$

$$+ \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \int_{D^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{x},\theta) d(vol)$$

$$+ \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \int_{\partial D_q^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \widetilde{\boldsymbol{q}}(\boldsymbol{x},\theta) d(area)$$

where Σ_e represents the summation over all the elements, $\boldsymbol{T}^e(\theta)$ denotes the nodal temperature vector of each element, $\boldsymbol{B}(\boldsymbol{x})$ is a matrix of the derivative of shape functions, $k(\boldsymbol{x},\theta)$ is the heat conductivity matrix, and $\boldsymbol{N}(\boldsymbol{x})$ is the shape function matrix.

Substituting eqn. (3.15) into (3.25), this yields to

$$\prod_p = \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \frac{1}{2} \left(\boldsymbol{K}_0^e + \sum_{i=1}^{N_k^*} \boldsymbol{\xi}_i^k(\theta)\, \boldsymbol{K}_i^e \right) \boldsymbol{T}^e(\theta) \qquad (3.26)$$

$$+ \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \left(\boldsymbol{F}_i^e\right) + \sum_e \left(\boldsymbol{T}^e(\theta)\right)^{\mathrm{T}} \widetilde{\boldsymbol{F}}^e(\theta)$$

in which the elemental coefficient matrices $\boldsymbol{K}_0^e$ and $\boldsymbol{K}_i^e$, the elemental volume-heat vector $\boldsymbol{F}_i^e$, and the elemental external-heat vector $\widetilde{\boldsymbol{F}}^e(\theta)$ are defined as follows:

$$\boldsymbol{K}_0^e = \int_{D^e} \left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} k_0(\boldsymbol{x}) \boldsymbol{B}(\boldsymbol{x}) d(vol) \qquad (3.27)$$

$$\boldsymbol{K}_i^e = \sqrt{\lambda_i^k}\int_{D^e}\left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}} k_i(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})d(vol) \quad i = 1,2,\cdots,N_k^* \tag{3.28}$$

$$\boldsymbol{F}_i^e = \int_{D^e}\left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{x},\theta)d(vol) \tag{3.29}$$

$$\widetilde{\boldsymbol{F}}^e(\theta) = \int_{\partial D_q^e}\left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}}\widetilde{\boldsymbol{q}}_n(\boldsymbol{x},\theta)d(area) \tag{3.30}$$

Minimizing the total potential $\prod_p$ by taking variation with respect to the temperature on both sides of Eq. (3.26) and equating to zero, then the following stochastic system of linear algebraic equations hold

$$\left(\boldsymbol{K}_0 + \sum_{i=1}^{N_k^*}\xi_i^k(\theta)\boldsymbol{K}_i\right)\boldsymbol{T}(\theta) = -(\boldsymbol{F}_i + \widetilde{\boldsymbol{F}}(\theta)) \tag{3.31}$$

in which the global coefficient matrices $\boldsymbol{K}_0$ and $\boldsymbol{K}_i$ are assembled respectively from elemental $\boldsymbol{K}_0^e$ and $\boldsymbol{K}_i^e$; the global volume-heat vector $\boldsymbol{F}_i$ is assembled from elemental $\boldsymbol{F}_i^e$; the global external-heat vector $\widetilde{\boldsymbol{F}}(\theta)$ is assembled from elemental $\widetilde{\boldsymbol{F}}^e(\theta)$; and $\boldsymbol{T}(\theta)$ is the unknown nodal temperature vector.

3.3.3 Generalized Stochastic Linear System

Following the finite element discretization of the SPDE systems for elastostatics and steady-state heat conduction above, it is observed that a complete mathematical analogy of the two problems exists. Hence, the resulting finite element formulations (3.24) and (3.31) can be expressed as a generalized stochastic system of linear algebraic equations

$$\left(\mathbf{K}_0 + \sum_{i=1}^{N_{C\ or\ k}^*}\boldsymbol{\xi}_i^{C\ or\ k}(\theta)\,\mathbf{K}_i\right)\boldsymbol{\Phi}(\theta) = \mathbf{F} + \tilde{\mathbf{F}} \tag{3.32}$$

in which the corresponding terms are expressed as

$$\mathbf{K}_0 = \sum\boldsymbol{K}_0^e = \sum\int_{D^e}\left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}}\left(D_0(\boldsymbol{x})\ or\ k_0(\boldsymbol{x})\right)\boldsymbol{B}(\boldsymbol{x})d(vol) \tag{3.33}$$

$$\mathbf{K}_i = \sum\boldsymbol{K}_i^e = \sum\sqrt{\lambda_i^{C\ or\ k}}\int_{D^e}\left(\boldsymbol{B}(\boldsymbol{x})\right)^{\mathrm{T}}(D_i(\boldsymbol{x})\ or\ (k_i(\boldsymbol{x}))\boldsymbol{B}(\boldsymbol{x})d(vol)$$

$$i = 1,2,\cdots,N_{C\ or\ k}^* \tag{3.34}$$

$$\mathbf{F} = \sum \boldsymbol{F}^e = \sum \int_{D^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} \left(\left(\boldsymbol{g}\rho_0(\boldsymbol{x}) + \sqrt{\lambda_i^{\rho}}\, \boldsymbol{g}\rho_i(\boldsymbol{x}) \right) \; or \; \boldsymbol{Q}(\boldsymbol{x},\theta) \right) d(vol)$$

$$i = 1,2,\cdots,N_{\rho}^{*} \tag{3.35}$$

$$\tilde{\mathbf{F}} = \sum \tilde{\boldsymbol{F}}^e = \sum \int_{\partial D_{\sigma \; or \; q}^e} \left(\boldsymbol{N}(\boldsymbol{x})\right)^{\mathrm{T}} (\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},\theta) \; or \; \tilde{\boldsymbol{q}}_n(\boldsymbol{x},\theta)) d(area) \tag{3.36}$$

$$\boldsymbol{\Phi} = \left(\boldsymbol{U}(\theta) \text{ or } \boldsymbol{T}(\theta)\right) \tag{3.37}$$

From the above expressions, the $\sum$ represents the necessary assembly over all elements, the matrix $\boldsymbol{K}_0$ is real, symmetric and positive-definite, the matrices $\boldsymbol{K}_i$ are all real and symmetric, and the random matrix sum $\boldsymbol{K}_0 + \sum_{i=1}^{N_{C \; or \; k}^{*}} \boldsymbol{\xi}_i^{C \; or \; k}(\theta)\, \boldsymbol{K}_i$ is, in the context of probability, real, symmetric and positive-definite. The terms inside the parenthesis with "or" symbol represent the elastostatics for the first term and the steady-state heat conduction for the second term. In addition, the boundary conditions $\tilde{\boldsymbol{u}}(\boldsymbol{x},\theta)$ and $\tilde{\boldsymbol{T}}(\boldsymbol{x},\theta)$ of the SPDE systems can be directly introduced into the generalized stochastic linear algebraic system (3.32).

The random coefficient matrix sum of eqn. (3.32) is almost surely real, symmetric and positive-definite such that

$$P\left\{ \det\left(\left(\boldsymbol{K}_0 + \sum_{i=1}^{N_{C \; or \; k}^{*}} \boldsymbol{\xi}_i^{C \; or \; k}(\theta)\, \boldsymbol{K}_i \right) \right) \neq 0 \right\} = 1 \tag{3.38}$$

Accordingly, the generalized stochastic system of linear algebraic equations (3.32) is always well-defined and the random displacement $\boldsymbol{U}(\theta)$ and temperature $\boldsymbol{T}(\theta)$ solution exists with probability one.

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Chapter 4

Solution Strategy for the Stochastic System of Linear Algebraic Equations

From the complete SPDE systems for elastostatics and steady-state heat conduction problems of random media in Chapter 3, the stochastic system of linear algebraic equations was derived. It is rewritten here for clarity:

$$\left(\mathbf{K}_0 + \sum_{i=1}^{N_C^{*\,or\,k}} \boldsymbol{\xi}_i^{C\,or\,k}(\theta)\,\mathbf{K}_i\right)\boldsymbol{\Phi}(\theta) = \mathbf{F} + \tilde{\mathbf{F}} \qquad (4.1)$$

The mathematical form of the stochastic linear algebraic system (4.1) is well-known in SFEM research. Hence, a number of methods have been developed already for the solution of these equations, namely: the Monte Carlo method [4.1-4.2]; the Neumann expansion method [4.3-4.4]; the polynomial chaos expansion method [4.5-4.7]; the perturbation method [4.8-4.9]; and recently, the joint diagonalization method [4.13]. From a computational point of view, the first four of these methods have encountered serious obstacles in practical implementations. In large and realistic problems the developed methodologies are either cumbersome or computationally intensive. On the other hand, the joint diagonalization strategy is promising and an improved method that is easy to implement. Therefore, the aim of this chapter is to explore the formulation of the joint diagonalization strategy; and in addition, to present an attempted development of the joint tridiagonalization for the stochastic linear systems.

The notation in eqn. (4.1) will not be used in this chapter but instead the formulation is presented in a more simple expression in the form $(\alpha_1 \boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_2 + \cdots + \alpha_n \boldsymbol{A}_n)\boldsymbol{x} = \boldsymbol{b}$, where the coefficients α_i $(i = 1, \cdots, n)$ represent the random variables, the real symmetric deterministic matrices are represented by $\boldsymbol{A}_i$ $(i = 1, \cdots, n)$, the deterministic/random vector is $\boldsymbol{b}$ and the unknown random vector is $\boldsymbol{x}$.

4.1 Preliminary

The standard linear deterministic finite element equations take the form of linear algebraic equations expressed as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.2}$$

where the matrix $\boldsymbol{A}$, vector $\boldsymbol{x}$ and vector $\boldsymbol{b}$ have different physical meanings depending on the system under consideration. In this thesis for example, the problems of elastostatics and steady-state heat conduction have different meanings for every term. Thus, for elastostatics problem, $\boldsymbol{A}$ represents the elastic stiffness matrix, $\boldsymbol{x}$ the unknown nodal displacement vector and $\boldsymbol{b}$ the nodal load vector. In the same way, it is the thermal conductivity matrix, the unknown nodal temperature vector and the nodal temperature-load vector, respectively for steady-state heat conduction problem.

The deterministic physical model depicted by eqn. (4.2) is not often valid for real structures composed of random media. This is due to the fact that real structures inherent various uncertainties of the design parameters such as material properties, geometry and/or loading conditions. In this case, the SFEM is used to capture the randomness of these design parameters. Using the SFEM formulation, it can be shown in Chapter 3 that the SPDEs of any physical problem can be reduced to the so called stochastic system of linear algebraic equations and takes the following simple form

$$(\alpha_1 \boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_2 + \cdots + \alpha_n \boldsymbol{A}_n)\boldsymbol{x} = \boldsymbol{b} \tag{4.3}$$

where the deterministic $m \times m$ real symmetric matrices $\boldsymbol{A}_i$ $(i = 1, \cdots, n)$, the unknown random real vector $\boldsymbol{x}$ and deterministic real vector $\boldsymbol{b}$ (the uncertainties of these vector can

be ignored for the moment since it will not affect the whole system as it will be shown later) have basically the same physical meanings as their counterparts in eqn. (4.2); also the real scalars α_i $(i = 1, \cdots, n)$ are random factors used to capture the inherent randomness of the system. Obviously, eqn. (4.3) decoupled the stochastic nature of the physical problem into the deterministic and stochastic parts in a simple way.

4.2 Overview of Existing Solution Schemes

In the SFEM research, several techniques have been used to solve the stochastic system of linear algebraic equations (4.3), namely: the Monte Carlo method; the Neumann expansion method; the polynomial chaos expansion method; the perturbation method; and recently, the joint diagonalization method. An outline of the first-four solution schemes is as follows:

The Monte Carlo method is simple and conceivably the most flexible method. The method requires a large number of simulations in order to statistically converge to the correct solution. Hence, it is extremely computationally demanding especially for large and realistic problems where a large number of samples have to be computed. To overcome this difficulty, improvements [4.1-4.2] have been made. In this method, eqn. (4.3) becomes a standard deterministic system of linear algebraic equations.

The Neumann expansion method [4.3-4.4] splits the left-hand side coefficients of (4.3) as a sum of a deterministic and stochastic matrix. Then using the Neumann series to expand the coefficients and the solution $\boldsymbol{x}$ is represented as a series. The computational costs of the Neumann expansion method depend on the number of terms required in the resulting series. Unfortunately, to achieve a given accuracy it requires more terms in the series for large problems and could become even more expensive than the Monte Carlo method.

The polynomial chaos expansion method [4.5-4.7] is implemented using a Gaussian random variables and orthonormal multivariate Hermite polynomials [4.10]. The solution $\boldsymbol{x}$

can be represented as polynomial chaos expansion if α_i are mutually independent Gaussian random variables. The unknown vector-valued coefficients of this expansion can be solved through a Galerkin approach whose shape functions are provided by the multivariate Hermite polynomials. The method can only be used to solve equations consisting of Gaussian random variables. In addition, the computational costs and the complexity involved in the derivation of multivariate Hermite polynomials increase exponentially as the number of random variables grows.

In the perturbation method [4.8-4.9], all the random variables α_i are decomposed into deterministic part and random part. The random solution $\boldsymbol{x}$ is approximated by Taylor's series where the origin and coefficients involved are both unknown deterministic vectors. The method's order depends on the number of terms expanded in Taylor series. As a result, several procedures can be defined, i.e. zero-order perturbation, first-order perturbation, second-order perturbation and so on. Due to increasingly high complexity of the analytic derivations as well as computational costs, the applications of higher-order perturbations are rare.

Lastly, the idea of joint diagonalization strategy is to simultaneously transform the real symmetric matrices A_i using a sequence of orthogonal similarity transformation which gradually decreases the off-diagonal elements of the matrices. Then, the classical Jacobi method is modified to solve the resulting average eigenvalue problem. The strategy is simply an approximation except if all the matrices have exactly the same eigenstructure. The formulation and detailed discussion of the joint diagonalization is explained in the next section.

4.3 Joint Diagonalization Strategy

This section discusses the new solution strategy for the stochastic system of linear algebraic equations (4.3) called the joint diagonalization. The formulation of the scheme is discussed below followed by the Jacobi-like algorithm for the average eigenvalue problem.

4.3.1 Formulation

The strategy aims to obtain an explicit solution of $\boldsymbol{x}$ by inverting the matrix sum $(\alpha_1\boldsymbol{A}_1 + \alpha_2\boldsymbol{A}_2 + \cdots + \alpha_n\boldsymbol{A}_n)$ in terms of the random variables α_i $(i = 1, \cdots, n)$. In order to have a well-defined solution, the matrix sum must be non-singular almost surely, i.e.

$$P(|\alpha_1\boldsymbol{A}_1 + \alpha_2\boldsymbol{A}_2 + \cdots + \alpha_n\boldsymbol{A}_n| \neq 0) = 1 \tag{4.4}$$

The main idea here is to simultaneously diagonalize all the matrices $\boldsymbol{A}_i$ $(i = 1, \cdots, n)$ such that $\boldsymbol{B}^{-1}\boldsymbol{A}_i\boldsymbol{B} = \boldsymbol{\Lambda}_i = diag(\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{im})$ $(i = 1, \cdots, n)$, so that eqn. (4.3) can be transformed into:

$$\boldsymbol{B}(\alpha_1\boldsymbol{\Lambda}_1 + \alpha_2\boldsymbol{\Lambda}_2 + \cdots + \alpha_n\boldsymbol{\Lambda}_n)\boldsymbol{B}^{-1}\boldsymbol{x} = \boldsymbol{b} \tag{4.5}$$

where the matrix $\boldsymbol{B}$ is assume to exist and be an invertible matrix, $\boldsymbol{\Lambda}_i$ are diagonal matrices containing the eigenvalues λ_{ij} $(j = 1, \cdots, m)$ of the $m \times m$ real symmetric matrix $\boldsymbol{A}_i$.

From eqn. (4.5), the solution $\boldsymbol{x}$ can now be solved explicitly. After a sequence of mathematical manipulations, the solution $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x} = \boldsymbol{D}\left(\frac{1}{\sum_{i=1}^{n}\alpha_i\lambda_{i1}}, \frac{1}{\sum_{i=1}^{n}\alpha_i\lambda_{i2}}, \cdots, \frac{1}{\sum_{i=1}^{n}\alpha_i\lambda_{im}}\right)^{\mathrm{T}} \tag{4.6}$$

where

$$\boldsymbol{D} = \boldsymbol{B}diag(d_1, d_2, \cdots, d_m) \tag{4.7}$$

$$\boldsymbol{B}^{-1}\boldsymbol{b} = (d_1, d_2, \cdots, d_m)^{\mathrm{T}} \tag{4.8}$$

After solving $\boldsymbol{x}$, the associated joint probability distribution and statistical moments, such as the expectation and covariance can be readily calculated. The strategy above is basically an average eigenvalue problem. In order to solve for $\boldsymbol{x}$, we need to obtain first the transform matrix $\boldsymbol{B}$ and the corresponding eigenvalues λ_{ij} $(i = 1, \cdots, n;\ j = 1, \cdots, m)$. This will be tackled in detail in the subsequent subsection.

4.3.2 Jacobi-Like Scheme for the Average Eigenvalue Problem

In linear algebra, the eigenvalue problem for square matrices can always be transformed into a real diagonal matrix using orthogonal operators. For example, in $m \times m$ real symmetric matrix $\boldsymbol{A}$ there exists an orthogonal matrix $\boldsymbol{Q}$, such that $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \cdots, \lambda_m)$, where $\boldsymbol{\Lambda}$ is a diagonal matrix and $\lambda_i \in \mathbb{R}\ (i = 1, \cdots, m)$. The diagonal elements of $\boldsymbol{\Lambda}$ are the eigenvalues of $\boldsymbol{A}$ and the columns of $\boldsymbol{Q}$ are the corresponding eigenvectors of $\boldsymbol{A}$.

The classical Jacobi method [4.11-4.12] is used to calculate the matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{Q}$. The object is to minimize the off-diagonal elements of the real symmetric matrix $\boldsymbol{A}$ so that the matrix becomes more nearly diagonal through a sequence of Givens similarity transformations. Even though Jacobi's idea is applied to a single matrix, the algorithm can be modified to accommodate multiple real symmetric matrices. The procedure is as follows:

Every Jacobi transformation gradually vanishes one pair of off-diagonal elements in real symmetric matrices $\boldsymbol{A}_k\ (k = 1, \cdots, n)$. In order to gradually vanish the pair of equal elements $(A_k)_{pq}$ and $(A_k)_{qp}$, the orthogonal Givens matrix $\boldsymbol{G}$ with the rotation angle β

$$G = G(p,q,\beta) \triangleq \begin{array}{c} \\ row\ p \\ \\ \\ row\ q \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} col\ p & & & & col\ q & \end{array} \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cos\beta & & \sin\beta & \\ & & & 1 & & \\ & & -\sin\beta & & \cos\beta & \\ & & & & & 1 \end{bmatrix} \end{array} \tag{4.9}$$

is employed. The series of orthogonal similarity transformations gradually reduce the sum of off-diagonal elements $\sum_{k=1}^{n} off(\boldsymbol{A}_k)$ where the off-diagonal elements is defined as $off(\boldsymbol{A}_k) \triangleq \sum_{i=1}^{m}\sum_{j=1, j\neq i}^{m}(\boldsymbol{A}_k)_{ij}^2$, $(k = 1, \cdots, n)$, which have no effect on the Frobenius norms defined as $\|\boldsymbol{A}_k\|_F \triangleq \left(\sum_{i=1}^{m}\sum_{j=1}^{m}(\boldsymbol{A}_k)_{ij}^2\right)^{1/2}$, $(k = 1, \cdots, n)$. For each transformation, the new matrices are calculated as $\boldsymbol{A}_k^* = \boldsymbol{G}(p,q,\beta)\boldsymbol{A}_k\boldsymbol{G}^{-1}(p,q,\beta)$ where only the rows p and q, and columns p and q are changed. After every transformation, with some algebraic manipulations the following equalities will hold

$$\sum_{k=1}^{n} off(\boldsymbol{A}_k^*) = \sum_{k=1}^{n} off(\boldsymbol{A}_k) - \sum_{k=1}^{n} 2(\boldsymbol{A}_k)_{pq}^2 + \sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2 \quad (4.10)$$

Hence, the minimization of the off-diagonal elements $\sum_{k=1}^{n} off(\boldsymbol{A}_k^*)$ is equivalent to minimizing $\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2$.

From the results of the new matrices $\boldsymbol{A}_k^*$, the expression $\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2$ is given as

$$\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2 = \sum_{k=1}^{n} 2\left(\frac{1}{2}\left((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\right)sin2\beta + (\boldsymbol{A}_k)_{pq}cos2\beta\right)^2$$

$$= (cos2\beta \;\; sin2\beta)\boldsymbol{J}\begin{pmatrix} cos2\beta \\ sin2\beta \end{pmatrix} \quad (4.11)$$

where $\boldsymbol{J} = \sum_{k=1}^{n}\begin{pmatrix} 2(\boldsymbol{A}_k)_{pq}^2 & (\boldsymbol{A}_k)_{pq}\left((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\right) \\ (\boldsymbol{A}_k)_{pq}\left((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\right) & \frac{1}{2}\left((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\right)^2 \end{pmatrix}$. (4.12)

Suppose that $e_1^{(J)}$ and $e_2^{(J)}$ denote the unit eigenvectors of $\boldsymbol{J}$, and $\lambda_1^{(J)} \geq \lambda_2^{(J)} \geq 0$ the eigenvalues of $\boldsymbol{J}$, then based on the theory of quadratic form the maximum and minimum eigenvalues are reached when $(cos2\beta \;\; sin2\beta)^T$ is equal to $e_1^{(J)}$ and $e_2^{(J)}$, respectively. Therefore, $\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2$ is minimized using the optimal Givens rotation angle β_{opt} based on the minimum eigenvalue as determine by

$$(cos2\beta \;\; sin2\beta)^{\mathrm{T}} = e_2^{(J)} \quad \left(cos2\beta_{opt} \geq 0\right) \quad (4.13)$$

for which the corresponding Givens matrix is formed.

The modified classical Jacobi algorithm is summarized in steps as follows:

Step 1. For all the entries of matrices $\boldsymbol{A}_k$ look for an entry (p, q) where $p \neq q$ such that $\sum_{k=1}^{n} 2(\boldsymbol{A}_k)_{pq}^2 \neq 0$.

Step 2. For every entry (p, q) that satisfies step 1, the Givens matrix $\boldsymbol{G}(p, q, \beta_{opt})$ can be formed by calculating first the optimal Givens rotation angle β_{opt} from eqn. (4.13).

Step 3. Update all the matrices $\boldsymbol{A}_k$ as $\boldsymbol{A}_k^* = \boldsymbol{G}(p, q, \beta_{opt})\boldsymbol{A}_k\boldsymbol{G}^{-1}(p, q, \beta_{opt})$. Note: only the p-th and q-th rows/columns in these matrices need to be updated.

Step 4. Repeat steps 1-3 until the process converges.

If the above steps has been performed K times, then the transform matrix $\boldsymbol{B}$ in eqn. (4.5) is computed as

$$\boldsymbol{B} = \boldsymbol{G}_1^{-1}\boldsymbol{G}_2^{-1}\cdots\boldsymbol{G}_K^{-1} = \boldsymbol{G}_1^{T}\boldsymbol{G}_2^{T}\cdots\boldsymbol{G}_K^{T} \tag{4.14}$$

and the diagonal entries in the final matrices $\boldsymbol{A}_k^*$ correspond to the eigenvalues λ_{ij} $(i = 1,\cdots,n;\ j = 1,\cdots,m)$. The convergence of this average eigenstructure problem is guaranteed by the Jacobi-like joint diagonalization algorithm since it can be shown that it is monotonously decreasing in the iterative procedure [4.13].

4.3.3 Discussions

- The Jacobi-like algorithm for multiple real symmetric matrices reduces to the classical Jacobi algorithm for a single real symmetric matrix and the joint diagonalization solution strategy gives the exact solution of $\boldsymbol{x}$ in this simple case, i.e.

$$\alpha_1\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{b} \tag{4.15}$$

$$\boldsymbol{x} = \frac{1}{\alpha_1}\boldsymbol{A}_1^{-1}\boldsymbol{b} \tag{4.16}$$

  Nevertheless, the present methods, such as the Monte Carlo method, the Neumann expansion method, the polynomial chaos method and the perturbation method do not have the above characteristic. Furthermore, the solution for the deterministic equation system (4.2) can be regarded as a special case of the solver for the more general stochastic equation system (4.3).

- Except if all the real symmetric matrices $\boldsymbol{A}_i$ $(i = 1,\cdots,n)$ have exactly the same eigenstructure, the joint diagonalization of eqn. (4.3) for $n \geq 2$ can only be approximately achieved. The approximate result is essentially an average eigenstructure that minimizes all the off-diagonal entries measured by $\sum_{k=1}^{n} off(\boldsymbol{A}_k)$. Consequently, the degree of the eigenstructure similarity of matrices dictates the effectiveness and efficiency of the approach.

- The right-hand side vector $\boldsymbol{b}$ of eqn. (4.3) has been assumed deterministic. In the present approach there is basically no intermediate operation required on $\boldsymbol{b}$ until the final solution $\boldsymbol{x}$ is calculated for the given random variables $\alpha_i\ (i = 1, \cdots, n)$ as affirmed in the explicit solution (4.6). Thus, the performance of the solution strategy is not influenced by the type of random variations.
- The Jacobi-like joint diagonalization procedure contributes the main computational cost which is proportional to the total number of matrices *n*.
- The Jacobi-like joint diagonalization is monotonously decreasing in the iterative procedure. A representative convergence history of the algorithm can be seen in [4.13].

4.4 Joint Tridiagonalization Strategy

The development of joint tridiagonalization strategy is somewhat similar in structure of the joint diagonalization strategy. The scheme transforms the real symmetric matrices $\boldsymbol{A}_i\ (i = 1, \cdots, n)$ into real symmetric tridiagonal matrices $\boldsymbol{T}_i$ as opposed to the diagonal matrices. The formulation of the scheme will be discussed next followed by the Jacobi-like algorithm for the resulting average eigenvalue problem.

4.4.1 Formulation

The key idea in this strategy is to simultaneously tridiagonalize all the $m \times m$ real symmetric matrices $\boldsymbol{A}_i\ (i = 1, \cdots, n)$ such that $\boldsymbol{C}^{-1}\boldsymbol{A}_i\boldsymbol{C} = \boldsymbol{T}_i\ \ (i = 1, \cdots, n)$, then eqn. (4.3) can be transformed into:

$$\boldsymbol{C}(\alpha_1\boldsymbol{T}_1 + \alpha_2\boldsymbol{T}_2 + \cdots + \alpha_n\boldsymbol{T}_n)\boldsymbol{C}^{-1}\boldsymbol{x} = \boldsymbol{b} \tag{4.17}$$

where the matrix $\boldsymbol{C}$ is assume to exist and be an invertible matrix and $\boldsymbol{T}_i\ \ (i = 1, \cdots, n)$ are real symmetric tridiagonal matrices and nonsingular. The solution $\boldsymbol{x}$ is given by

$$\boldsymbol{x} = \boldsymbol{C}\left(\sum_{i=1}^{n} \boldsymbol{\alpha}_i \boldsymbol{T}_i\right)^{-1}\boldsymbol{C}^{-1}\boldsymbol{b} \tag{4.18}$$

Hence, in order to get the solution $\boldsymbol{x}$ it is necessary to obtain first the transform matrix $\boldsymbol{C}$ and the corresponding tridiagonal matrices $\boldsymbol{T_i}$.

4.4.2 Jacobi-Like Scheme for the Average Eigenvalue Problem

In linear algebra, a square matrix can be transformed into a real symmetric tridiagonal matrix using a series of orthogonal similarity transformation. For example, in $m \times m$ real symmetric matrix $\boldsymbol{A}$ there exists an orthogonal matrix $\boldsymbol{P}$, so that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{T}$, where $\boldsymbol{T}$ is a real symmetric tridiagonal matrix having the same eigenvalues of $\boldsymbol{A}$ and the columns of $\boldsymbol{P}$ are the corresponding eigenvectors of $\boldsymbol{A}$.

Here, the classical Jacobi method [4.11-4.12] is used to calculate the matrices $\boldsymbol{T}$ and $\boldsymbol{P}$ but with some modifications. The aim here is to minimize the off-tridiagonal elements of the real symmetric matrix $\boldsymbol{A}$ so that the matrix becomes more nearly tridiagonal through a sequence of Givens similarity transformations.

The orthogonal Givens matrix $\boldsymbol{G}$ (4.9) is used to gradually diminish one pair of off-tridiagonal elements in real symmetric matrix $\boldsymbol{A}_k$ $(k = 1, \cdots, n)$. In each orthogonal similarity transformations the sum of off-tridiagonal elements $\sum_{k=1}^{n} offtri(\boldsymbol{A}_k)$ where the off-tridiagonal elements is defined as $offtri(\boldsymbol{A}_k) \triangleq \sum_{i=1}^{m-2}\sum_{j=i+2}^{m}(\boldsymbol{A}_k)_{ij}^2$, $(k = 1, \cdots, n)$, gradually diminishes in which each operation have no effect on the Frobenius norms of $\boldsymbol{A}_k$. The new matrices are computed as $\boldsymbol{A}_k^* = \boldsymbol{G}(p, q, \beta)\boldsymbol{A}_k\boldsymbol{G}^{-1}(p, q, \beta)$ where only the rows p and q, and columns p and q are changed. The following equalities hold after each transformation

$$\sum_{k=1}^{n} offtri(\boldsymbol{A}_k^*) = \sum_{k=1}^{n} off(\boldsymbol{A}_k) - \sum_{k=1}^{n} 2(\boldsymbol{A}_k)_{pq}^2 + \sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2 - \sum_{k=1}^{n}\sum_{i=1,}^{m-1}\sum_{j=i+1}^{m} 2(\boldsymbol{A}_k^*)_{ij}^2 \tag{4.19}$$

Thus, the minimization of $\sum_{k=1}^{n} offtri(\boldsymbol{A}_k^*)$ is equivalent to minimizing the expressions $\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2 - \sum_{k=1}^{n}\sum_{i=1,}^{m-1}\sum_{j=i+1}^{m} 2(\boldsymbol{A}_k^*)_{ij}^2$.

From the results of the new matrices $\boldsymbol{A}_k^*$,

$$\sum_{k=1}^{n} 2(\boldsymbol{A}_k^*)_{pq}^2 - \sum_{k=1}^{n} \sum_{i=1,}^{m-1} \sum_{j=i+1}^{m} 2(\boldsymbol{A}_k^*)_{ij}^2 = (cos2\beta \;\; sin2\beta)\boldsymbol{J}_1 \begin{pmatrix} cos2\beta \\ sin2\beta \end{pmatrix} - 2(cos\beta \;\; sin\beta)\boldsymbol{J}_2 \begin{pmatrix} cos\beta \\ sin\beta \end{pmatrix} \quad (4.20)$$

where

$$\boldsymbol{J}_1 = \sum_{k=1}^{n} \begin{pmatrix} 2(\boldsymbol{A}_k)_{pq}^2 & (\boldsymbol{A}_k)_{pq}\big((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\big) \\ (\boldsymbol{A}_k)_{pq}\big((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\big) & \frac{1}{2}\big((\boldsymbol{A}_k)_{qq} - (\boldsymbol{A}_k)_{pp}\big)^2 \end{pmatrix} \quad (4.21)$$

$$\boldsymbol{J}_2 = \sum_{k=1}^{n} \begin{pmatrix} (\boldsymbol{J}_2)_{\mathbf{11}} & (\boldsymbol{J}_2)_{\mathbf{12}} \\ (\boldsymbol{J}_2)_{\mathbf{21}} & (\boldsymbol{J}_2)_{\mathbf{22}} \end{pmatrix} \quad (4.22)$$

$$(\boldsymbol{J}_2)_{\mathbf{11}} = (\boldsymbol{A}_k)_{p-1,p}^2 + (\boldsymbol{A}_k)_{p+1,p}^2 + (\boldsymbol{A}_k)_{q-1,q}^2 + (\boldsymbol{A}_k)_{q+1,q}^2 \quad (4.23)$$

$$(\boldsymbol{J}_2)_{\mathbf{12}} = (\boldsymbol{J}_2)_{\mathbf{21}} = (\boldsymbol{A}_k)_{p-1,p}(\boldsymbol{A}_k)_{p-1,q} + (\boldsymbol{A}_k)_{p+1,p}(\boldsymbol{A}_k)_{p+1,q} - (\boldsymbol{A}_k)_{q-1,p}(\boldsymbol{A}_k)_{q-1,q} - (\boldsymbol{A}_k)_{q+1,p}(\boldsymbol{A}_k)_{q+1,q} \quad (4.24)$$

$$(\boldsymbol{J}_2)_{\mathbf{22}} = (\boldsymbol{A}_k)_{p-1,q}^2 + (\boldsymbol{A}_k)_{p+1,q}^2 + (\boldsymbol{A}_k)_{q-1,p}^2 + (\boldsymbol{A}_k)_{q+1,p}^2 \quad (4.25)$$

Then based on the optimization theory, the optimal Givens rotation angle β_{opt} is computed numerically using the Newton-Raphson method from eqn. (4.20), so that the corresponding Givens matrix is formed.

The modified classical Jacobi algorithm here is the same as in the joint diagonalization strategy except for some minor alterations. The procedure is as follows:

Step 1. For all the entries of matrices $\boldsymbol{A}_k$ look for an entry (p, q) where $q = p + 2$ and $p < m - 1$ such that $\sum_{k=1}^{n} 2(\boldsymbol{A}_k)_{pq}^2 \neq 0$.

Step 2. For every entry (p, q) that satisfies step 1, the Givens matrix $\boldsymbol{G}(p, q, \beta_{opt})$ can be formed by calculating first the optimal Givens rotation angle β_{opt} from eqn. (4.20) using Newton-Raphson method.

Step 3. Update all the matrices $\boldsymbol{A}_k$ as $\boldsymbol{A}_k^* = \boldsymbol{G}(p, q, \beta_{opt})\boldsymbol{A}_k\boldsymbol{G}^{-1}(p, q, \beta_{opt})$.

Note: only the p-th and q-th rows/columns in these matrices need to be updated.

Step 4. Repeat steps 1-3 until the process converges.

If the above steps has been performed K times with Givens matrices $\boldsymbol{G}_1, \boldsymbol{G}_2, \cdots, \boldsymbol{G}_K$ respectively, then the transform matrix $\boldsymbol{C}$ in eqn. (4.18) is given by

$$\boldsymbol{C} = \boldsymbol{G}_1^{-1}\boldsymbol{G}_2^{-1}\cdots\boldsymbol{G}_K^{-1} = \boldsymbol{G}_1^{T}\boldsymbol{G}_2^{T}\cdots\boldsymbol{G}_K^{T} \tag{4.26}$$

4.4.3 Discussions

The proposed joint tridiagonalization strategy above has the following issues:

- The proposed algorithm reduces to a single real symmetric matrix and the proposed joint tridiagonalization solution strategy gives the exact solution of $\boldsymbol{x}$ as

$$\boldsymbol{x} = \boldsymbol{C}(\boldsymbol{\alpha}_1\boldsymbol{T}_1)^{-1}\boldsymbol{C}^{-1}\boldsymbol{b} \tag{4.27}$$

- The joint tridiagonalization strategy is only an approximate except if all the symmetric matrices $\boldsymbol{A}_i$ $(i = 1, \cdots, n)$ have exactly the same eigenstructure.
- For a single random variable, the convergence of the scheme is guaranteed. For random variables greater than 2, i.e. $n \geq 2$ the scheme does work for some real symmetric matrices but for most matrices the strategy is not converging properly. The author believes that the strategy will work to all real symmetric matrices. The algorithm was implemented in MATLAB. The errors might be in the code itself or the formulations. Due to time constraint the development was not completed, it is therefore recommended that this algorithm can be further verified in the future studies.

4.5 Summary of the Joint Diagonalization Approach

The response statistics for any static stochastic systems can be achieved by following the procedure below:

1) Discretize the random material parameters using the explicit F-K-L representation scheme;
2) Discretize the unknown field with finite element mesh in spatial dimension;

3) Using the standard finite element formulation procedure construct the stochastic system of linear algebraic equations by taking into consideration the F-K-L expansion of the random material parameters involved;
4) Approximate the joint diagonalization of all matrices using the steps outlined in Section 4.3.2; and
5) Obtain the response vector $\boldsymbol{u}$ for a specific realization of random variables.

Note that the joint diagonalization strategy is applicable to any real symmetric matrices. The procedure above is integrated into a SFEM framework and then used in the numerical examples of Chapter 5.

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Chapter 5
Numerical Examples

In this chapter, numerical results for steady-state heat conduction and elastostatics problems are presented to verify the joint diagonalization strategy and examine the significance of the uncertainties in the material model. Using the SFEM framework of [5.1], two separate analyses were prepared to differentiate the effect of randomness of the thermal conductivity and Young's modulus on the structure under consideration.

The numerical problem concerns a concrete wastewater pipe buried in ground. The cross-sectional area of the concrete pipe is shown in Figure 5.1 with inner radius of 0.6 m, outer radius of 0.8 m and thickness of 0.2 m. The concrete material is assumed to be isotropic and the deterministic material properties are Poisson's ratio $\nu = 0.2$, coefficient of thermal expansion $\alpha = 10^{-5}$/°C and density $\rho = 1500$ kg/m^3 which are assumed to be of constant values. The major concrete material properties such as thermal conductivity $k = k(x, \theta)$ and Young's modulus $E = E(x, \theta)$ are assumed to be random and approximately modeled as independent stationary Gaussian stochastic fields (Note: the joint diagonalization formulation is not restricted to any specific probability distribution of random variables which means any probability distribution can be used). The mean value of thermal conductivity is 1.7 W/(mK) and the corresponding covariance function is defined as $0.116e^{-\left(\frac{(x_1-x_2)^2+(y_1-y_2)^2}{0.2}\right)^{2/3}}$ W^2/(mK)^2. Analogously, the mean value of Young's modulus is 60 GPa and the corresponding covariance function is defined by the equation as

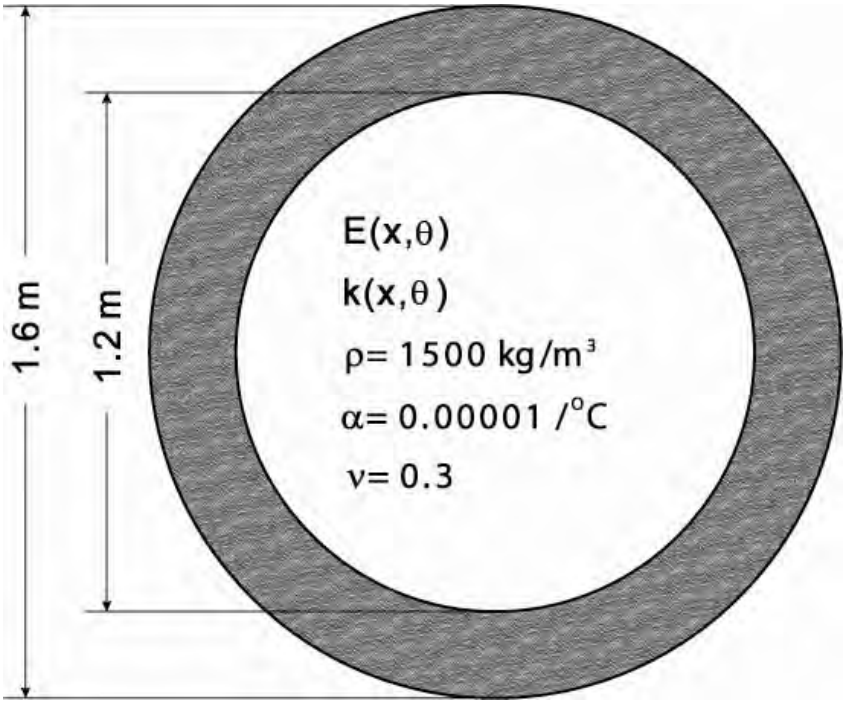

Figure 5.1 A cross section of concrete wastewater pipe.

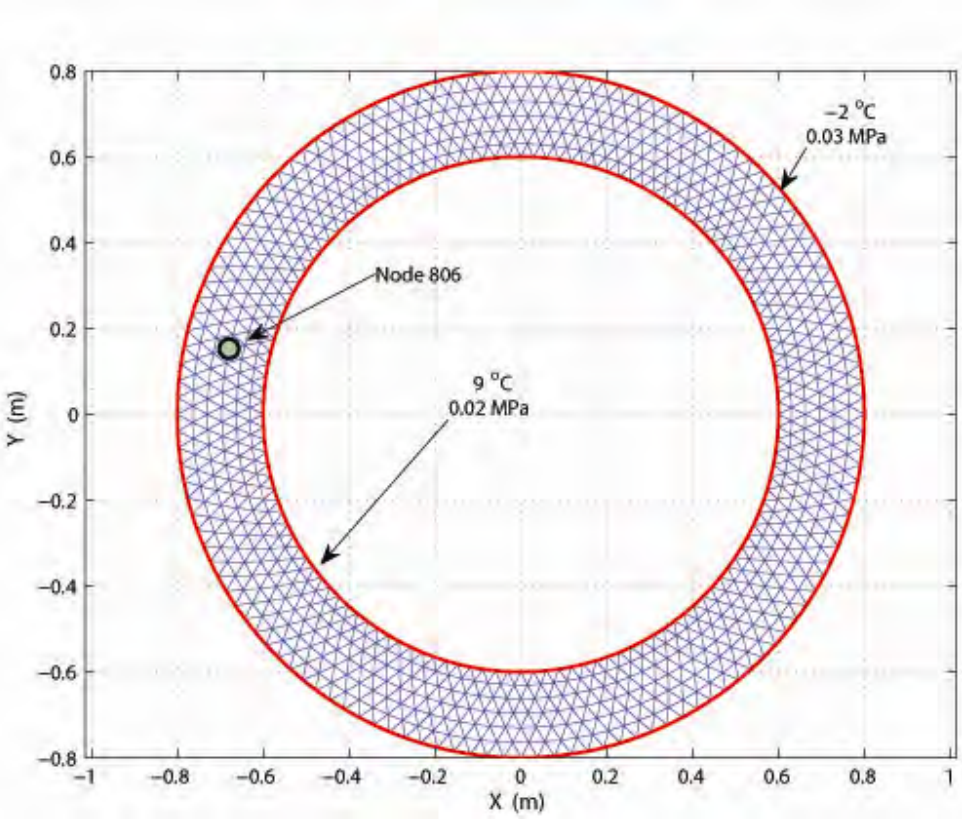

Figure 5.2 Finite element mesh and boundary conditions of the concrete wastewater pipe.

$144e^{-\left(\frac{(x_1-x_2)^2+(y_1-y_2)^2}{0.2}\right)^{2/3}}$ GPa^2.

The concrete pipe is subjected by two different types of loading, namely thermal and pressure loads both in the internal and external surfaces due to the wastewater flow inside and the surrounding earth outside. The applied temperatures are 9°C and -2°C at the inner and outer surfaces, respectively; and the applied pressures are 0.02 MPa and 0.03 MPa in the inner and outer surfaces, respectively.

Further, the geometry of the concrete wastewater pipe is meshed using triangular finite elements consisting of 896 nodes as shown in Figure 5.2.

In the following sections, the solutions of the pipe due to steady-state heat conduction and elastostatics are presented and discussed. Moreover, the joint diagonalization strategy which is integrated into the SFEM framework was used to generate the results in the next sections. In addition, the Monte Carlo method and Neumann expansion method were used to compare the effectiveness of the joint diagonalization method.

5.1 Example 1 – Steady-State Heat Conduction

First of all, the random thermal conductivity of the concrete pipe is explicitly expanded using F-K-L representation scheme. In order to determine the number of terms needed in the F-K-L expansion, the approximation error due to truncation is controlled within 10% in terms of the difference of variance. For a specific realization of the random thermal conductivity of concrete pipe see Figure 5.3. Then in space dimension, the unknown field is discretized with finite element mesh. Using the standard finite element formulation the problem is reduced to a stochastic system of algebraic linear equations consisting of 22 random variables.

As shown in Figure 5.3, the random thermal conductivity distribution varies significantly from the mean value of 1.7 W/(mK). In fact, it varies approximately by 15% and 27% from the minimum to maximum conductivity values, respectively. This means

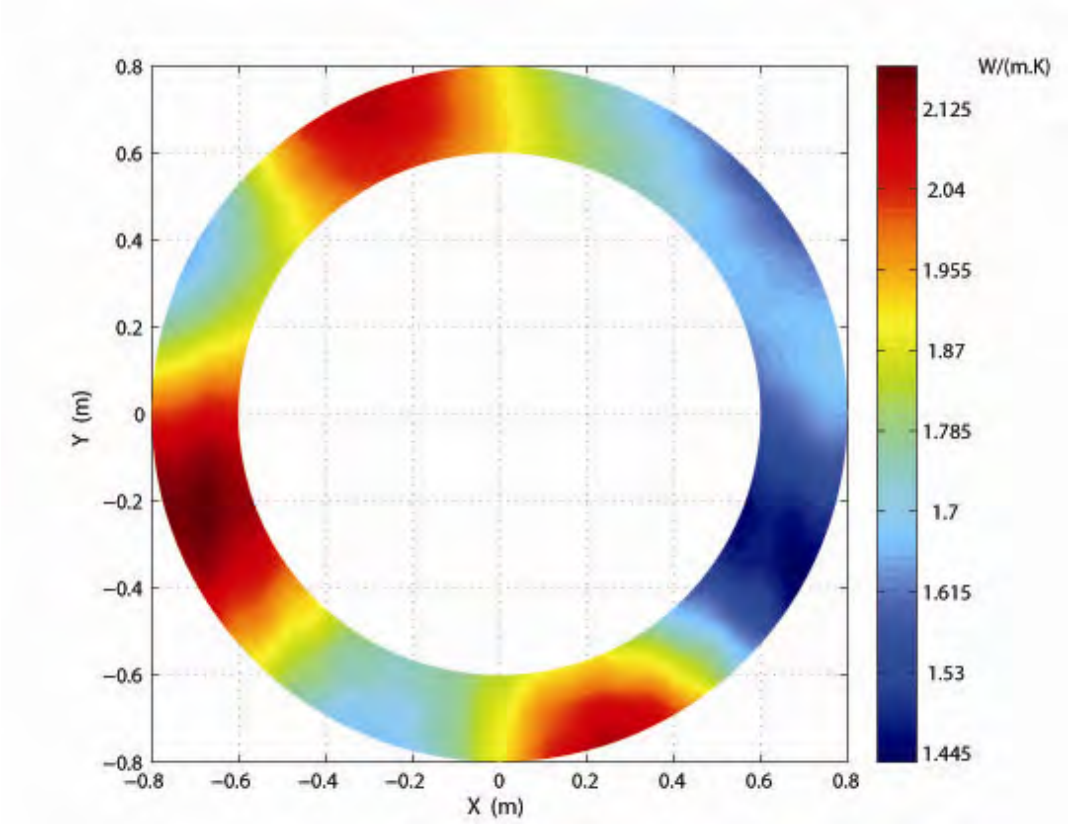

Figure 5.3 Thermal conductivity of the pipe reconstructed from F-K-L expansion.

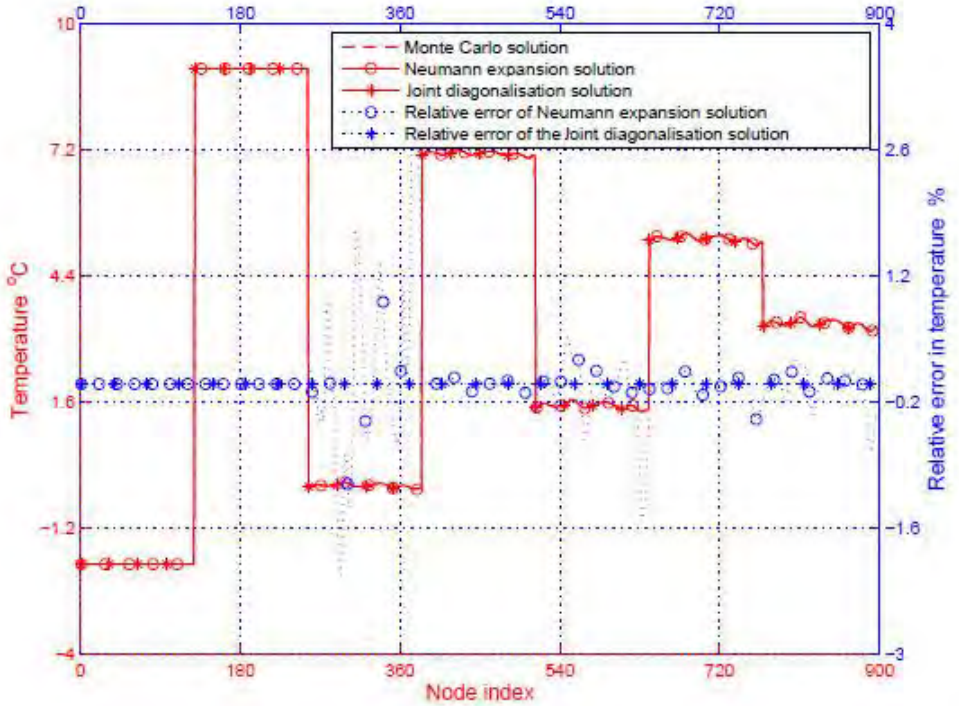

Figure 5.4 Comparison of the Monte Carlo solution, Neumann expansion solution and Joint diagonalization solution

that the randomness of the thermal conductivity might affect the structural response of the pipe inside-out.

In the heat conduction analysis, the stochastic system of linear algebraic equations (4.3) is composed of 23 – 896 × 896 real symmetric matrices including the mean matrix. The joint diagonalization method is then applied to compute a total number of 8960 sample solutions of the stochastic system of linear algebraic equations. In order to verify the accuracy of the joint diagonalization strategy, the samples above are compared with Monte Carlo solutions and Neumann expansion solutions. As shown in Figure 5.4 the joint diagonalization method agrees well with the Monte Carlo method. However, the Neumann expansion method is converging much slower compare to the joint diagonalization method. Hence, in terms of convergence rate the joint diagonalization method is much better than the Neumann expansion method.

Moreover, the random temperature distribution of the concrete pipe due to the inner and outer surfaces temperature boundary conditions of 9°C and -2°C, respectively is shown in Figure 5.5. Even though the variation of thermal conductivity is fairly significant as shown in Figure 5.3, Figure 5.5 reveals that the random variation of temperature distribution is rather small. The result is somewhat similar to the deterministic linearly distributed temperature from inner surfaces of the pipe towards the outer surfaces. The result is attributed to the strong fixed-temperature boundary conditions applied on both the inside and outside surfaces of the pipe. It shows that the randomness of the thermal conductivity does not really affect the distribution of temperature within the walls of the pipe. This occurrence is more apparent in Figure 5.6 where the temperature difference of the random solution and the deterministic solution where the thermal conductivity has a mean value of 1.7 W/mK was plotted. It reveals that the maximum variation of the temperature is about 0.15°C (less than 10%) only which occurs approximately at the center of the pipe wall.

To have an idea of the variation of temperature at any point of the wall, an inner node 806 (see Figure 5.2) approximately at the center of the pipe wall was selected. The probability distribution of temperature at the selected node as shown in Figure 5.7 was generated from a simple statistical counting of the sample solutions. The figure shows a

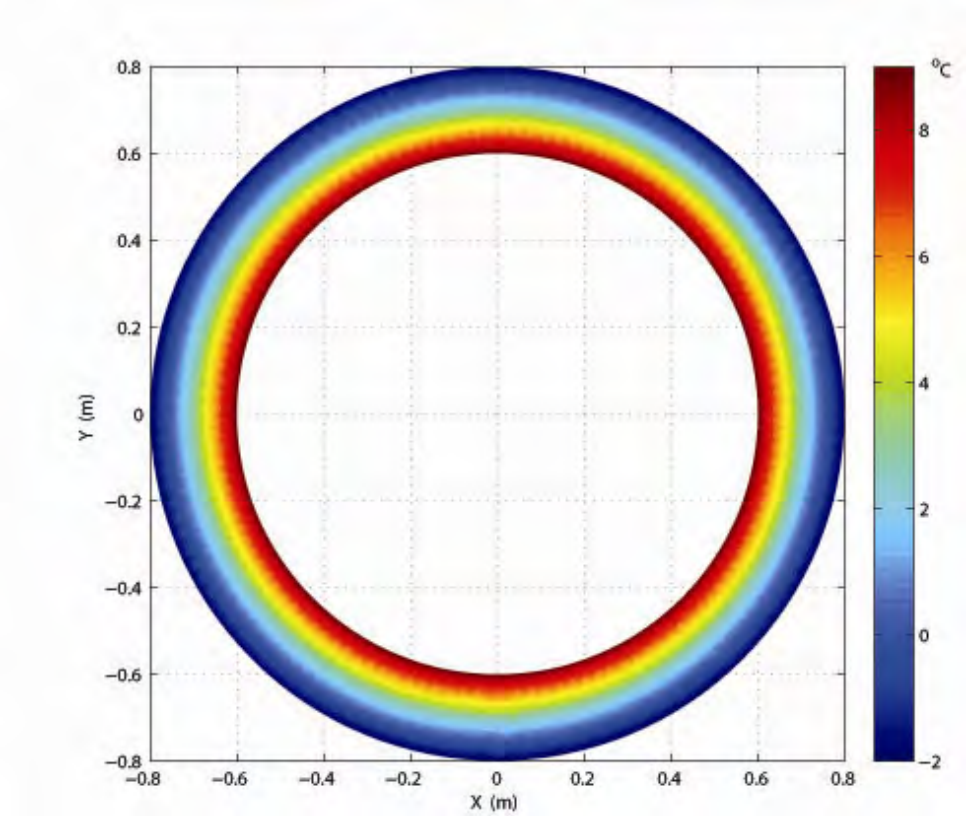

Figure 5.5 Random temperature distribution in the pipe.

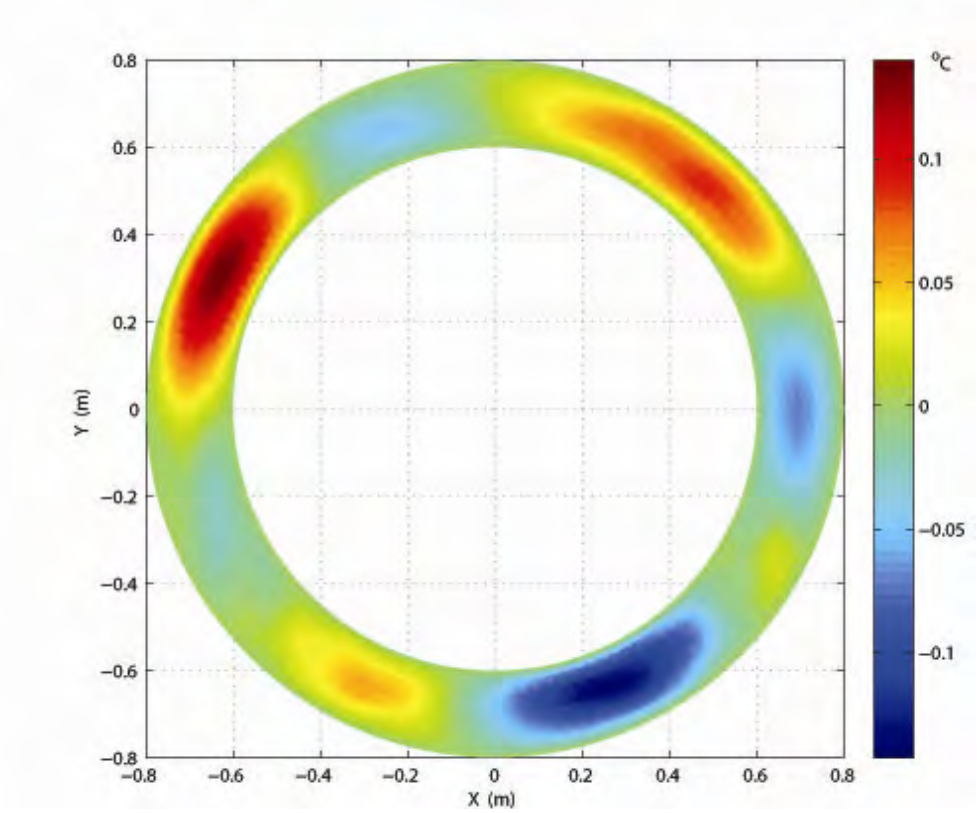

Figure 5.6 Temperature distribution difference between the random and mean-value

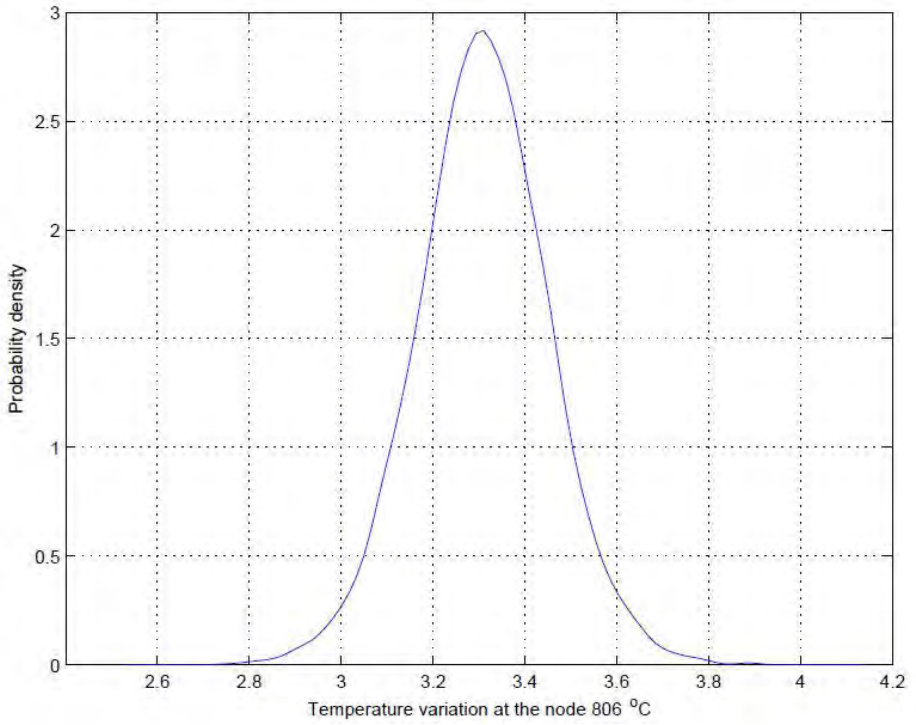

Figure 5.7 Temperature variation of the pipe at node 806.

Gaussian distribution of the temperature as confirmed by the assumption made above. This graph usually gives a better technical reference to design engineers.

5.2 Example 2 – Elastostatics

The concrete wastewater pipe was analyzed as 2D plane strain problem. For plane strain state, the longitudinal elements of the pipe are assumed to be in the z-direction for convenience, so that the cross section is in the xy-plane. The surface tractions and body forces are applied perpendicular to the longitudinal elements, independent of the z-direction and do not vary along the length. Moreover, it is assumed that all cross sections are in the same condition. Hence, it is sufficient to consider only a unit thickness of the pipe in the xy-plane.

In a plane strain analysis, the displacement in the z-direction must be constant or it can be assumed to be zero. The strains ε_z , γ_{xz} and γ_{yz} are taken as zero. Further, the stress components τ_{xz} and τ_{yz} are zero, and σ_z can be found from σ_x and σ_y. Thus, the

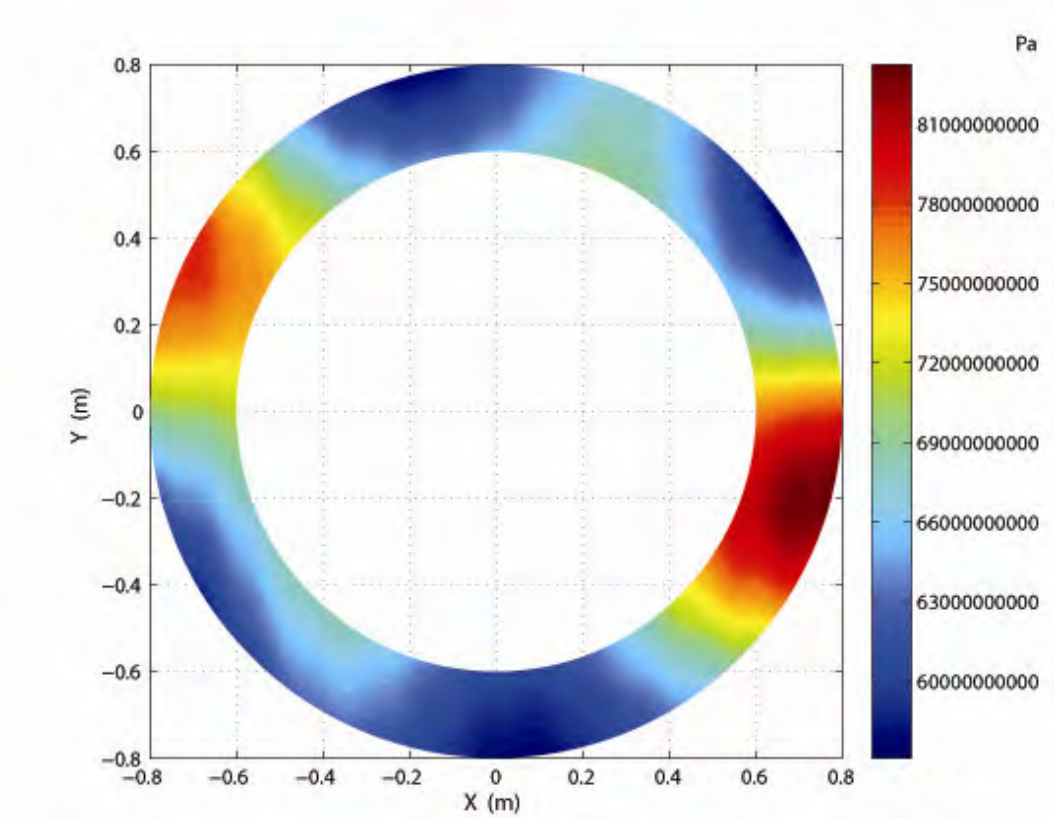

Figure 5.8 Young's modulus reconstructed from F-K-L expansion.

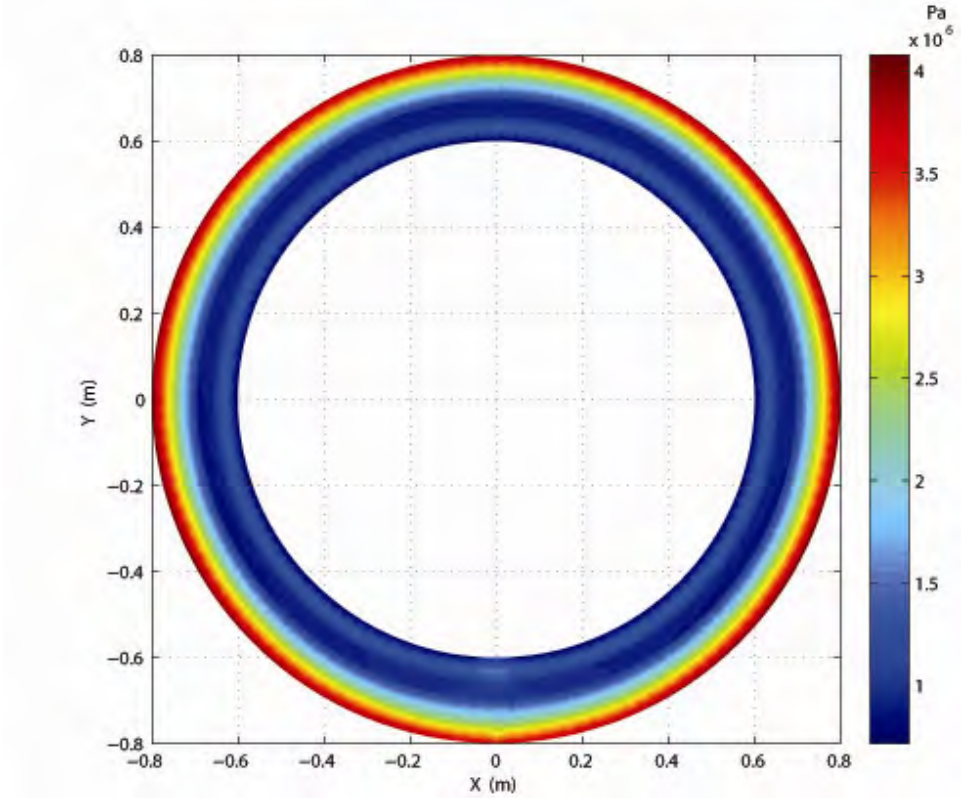

Figure 5.9 Mean-value distribution of the principal stress σ_1

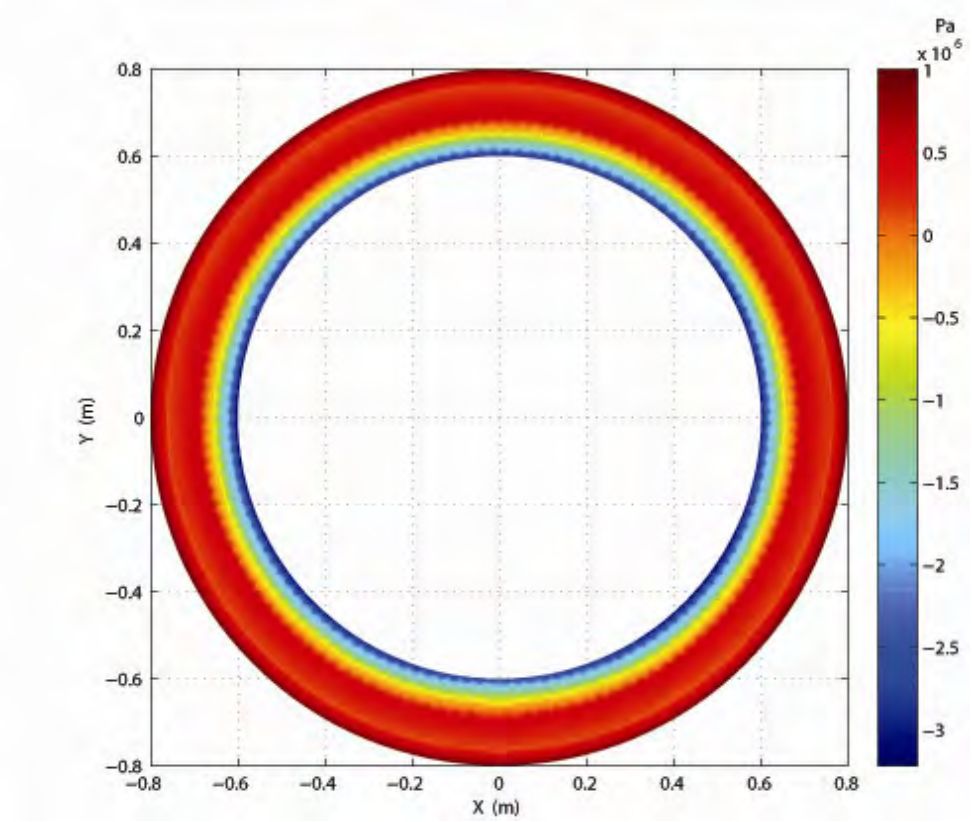

Figure 5.10 Mean-value distribution of the principal stress σ_2

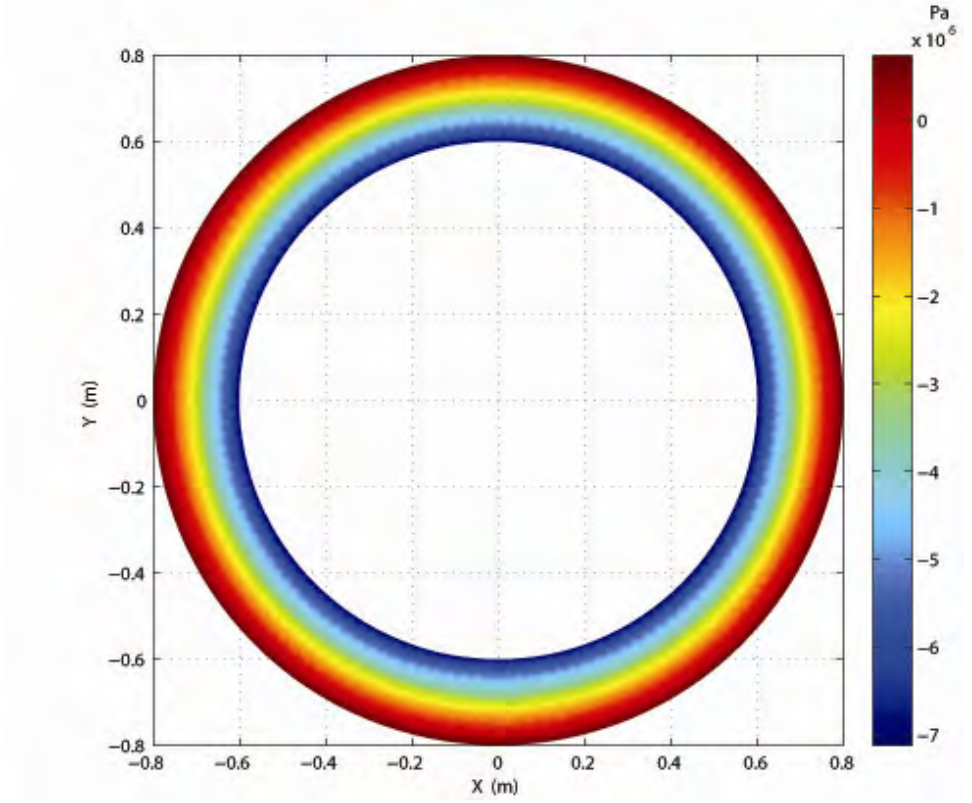

Figure 5.11 Mean-value distribution of the principal stress σ_3

plane strain problem reduces to the determination of σ_x, σ_y and τ_{xy} which are functions of x and y only. For a detailed discussion of plane strain problems see [5.2] for instance.

Following the steps in Section 5.1, the random Young's modulus of the concrete pipe was explicitly expanded using F-K-L representation of random media. For a specific realization of a random material model, Figure 5.8 shows a random Young's modulus reconstructed from F-K-L expansion. The random Young's modulus fluctuates appreciably from the mean value of 60 GPa. In fact, it varies approximately by 5% and 39% from the minimum to maximum Young's modulus values, respectively. Thus, the randomness of the material Young's modulus might affect the structural response of the pipe significantly.

Using the Young's modulus mean-value of 60 GPa, the distributions of the three principal stresses (σ_1, σ_2, σ_3) around the pipe due to the thermal and pressure loads were calculated and plotted as shown in Figures 5.9-5.11. For a particular solution, Figure 5.9 shows that the distribution of the 1st principal stress σ_1 had varied around the pipe from a minimum value of 0.725 MPa in the neighborhood of inner surfaces and midsection of the wall to a maximum value of 4.075 MPa in the vicinity of outer surfaces. Notice that the stresses are all positive. Also, as shown in Figure 5.10 the distribution of 2nd principal stress σ_2 have a minimum value of -3.2 MPa in the inner surfaces and increases uniformly to a maximum value of 1 MPa in the outer wall. Though the result shows that negative stresses have the maximum numerical values but positive stresses prevail around the pipe. Furthermore, the distribution of the 3rd principal stress σ_3 (see Figure 5.11) have a minimum value of -7.1 MPa in the inner surfaces and increases almost regularly towards the outside surfaces to a maximum value of 0.75 MPa in the outer surfaces. Clearly, the negative stresses dominated the distribution around the pipe. Accordingly, the maximum and minimum values of the three principal stresses had occurred at the outer surfaces and reduced towards the inner surfaces of the pipe, respectively. Moreover, the results are mostly uniformly distributed throughout the pipe for the three principal stresses.

Solving the problem by considering a random Young's modulus of the material, the random distribution of the three principal stresses (σ_1, σ_2, σ_3) are shown in Figures 5.12-5.14. The 1st principal stress σ_1 distribution as shown in Figure 5.12 had varied around the pipe from a minimum value of 0.625 MPa to a maximum value of 5.325 MPa. Notice that

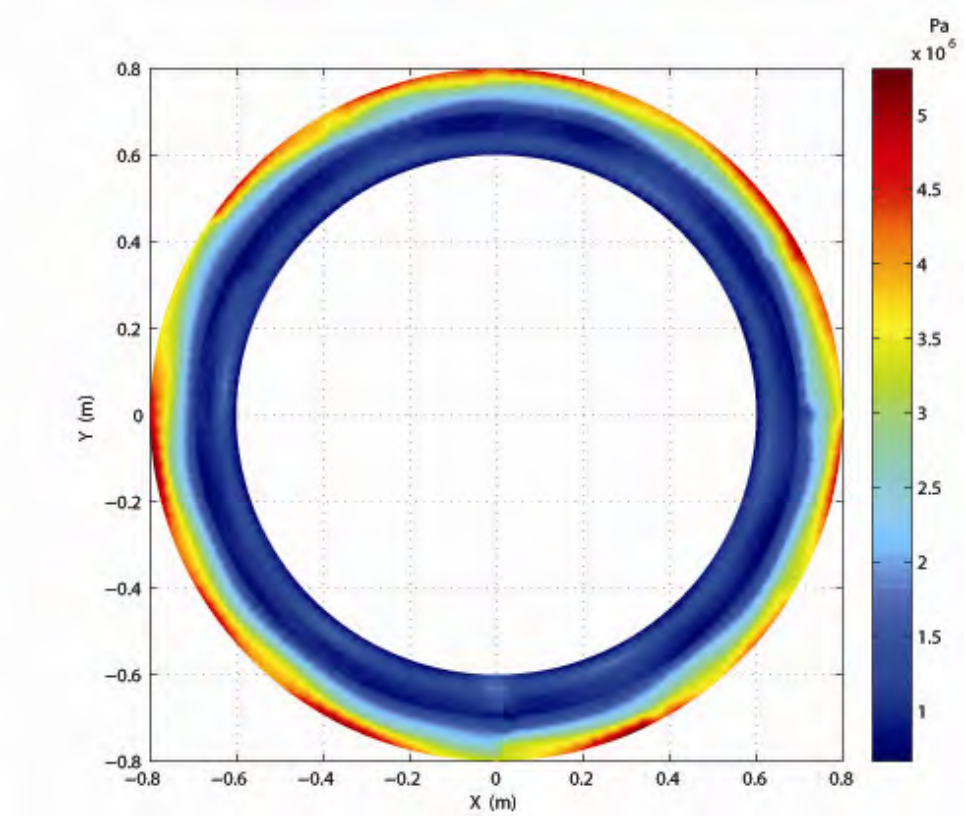

Figure 5.12 Distribution of principal stress σ_1 due to random Young's modulus

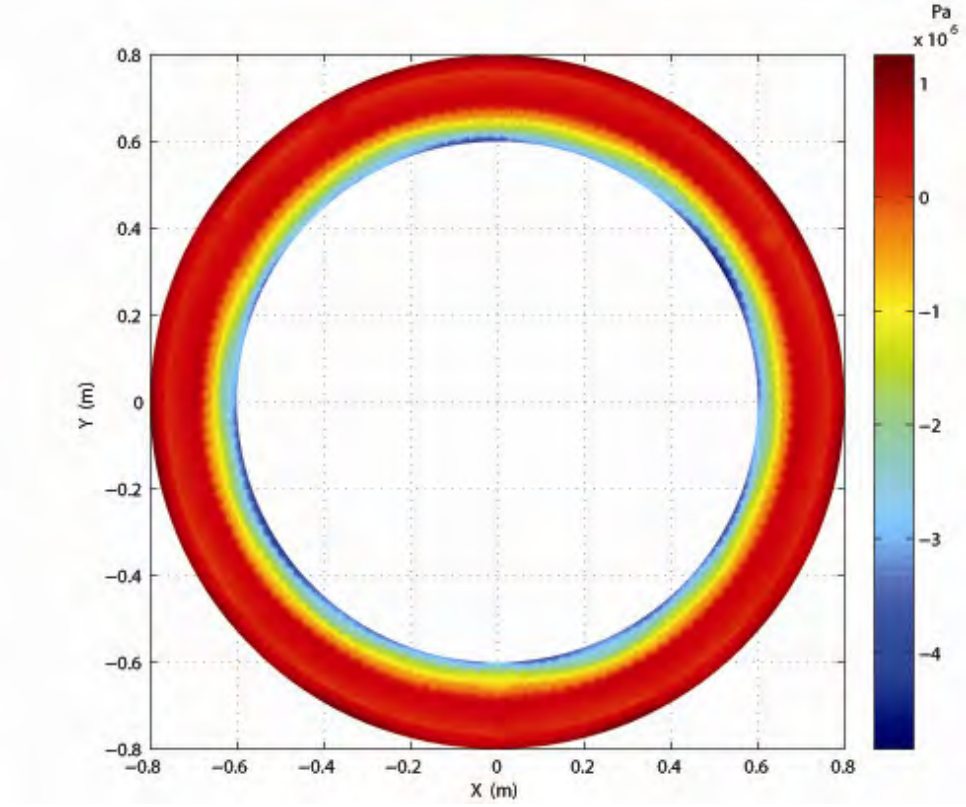

Figure 5.13 Distribution of principal stress σ_2 due to random Young's modulus

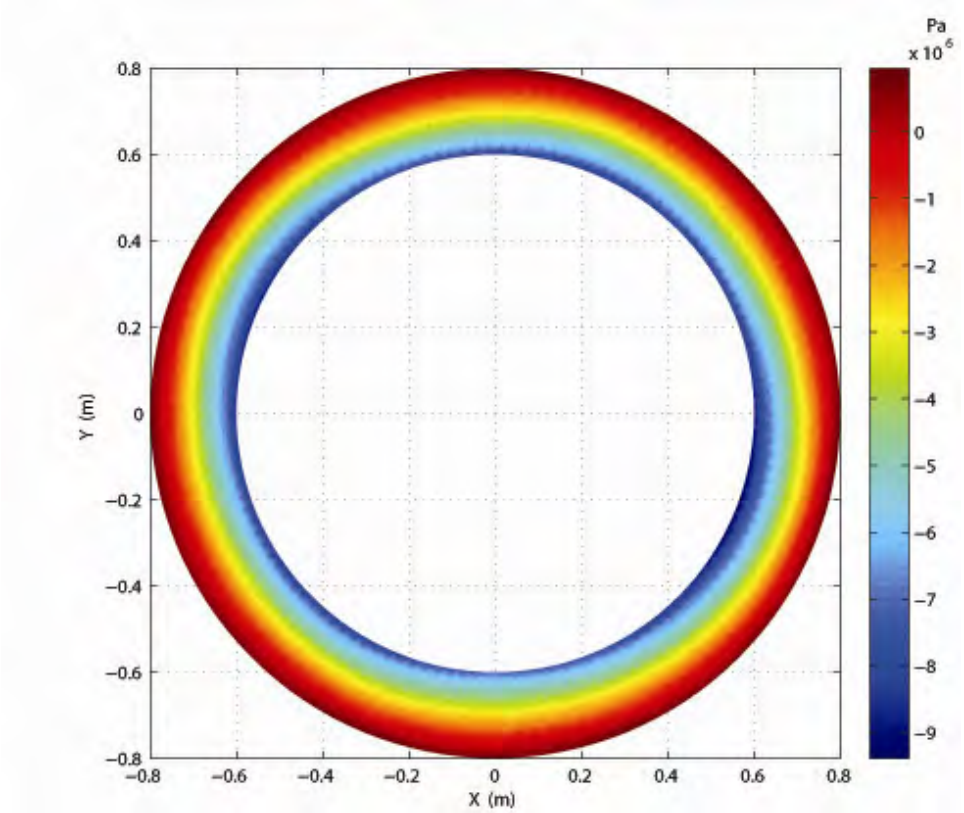

Figure 5.14 Distribution of principal stress σ_3 due to random Young's modulus

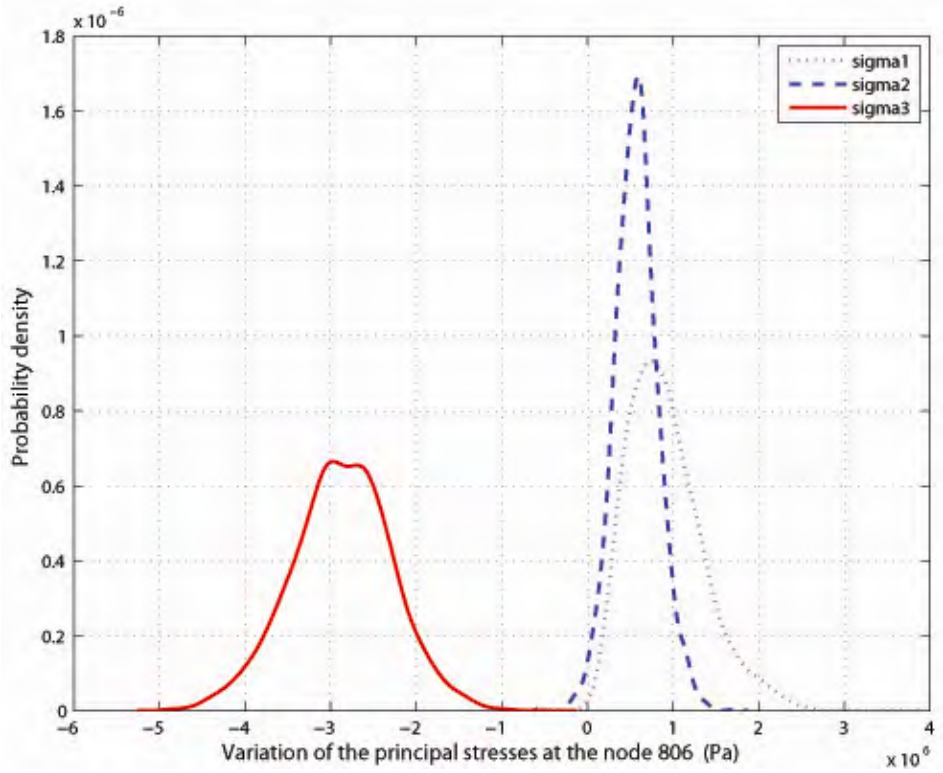

Figure 5.15 Variation of the principal stresses at node 806.

the stresses are all positive. The fluctuations of stress distribution near the vicinity of the outer surfaces are more pronounced then becomes lesser in the neighborhood of the inner surfaces. In addition, the distribution of 2nd principal stress σ_2 in Figure 5.13 had fluctuated in the inner region of the pipe from a minimum value of -4.85 MPa and almost uniform distribution towards the outer region with a maximum value of 1.25 MPa. The plot also shows that numerical values of negative stresses had prevailed in the pipe but the positive stresses spreads more that cover most of the region towards the outside surface. Furthermore, Figure 5.14 shows the 3rd principal stress σ_3 distribution have a minimum value of -9.4 MPa at the inner region to a maximum value of 1 MPa at the outside wall. Obviously, the negative stresses dominated the result. Likewise, the maximum and minimum values of principal stresses had occurred at the outer surfaces and reduced towards the inner surfaces of the pipe, respectively. Principal stress variations are more pronounced at the inner regions of the pipe. Moreover, the results for the 1st principal stress fluctuates significantly at the outer surfaces and moves towards the center of the pipe but distributes more uniformly near the wall center in the direction of the pipe center. The 2nd principal stresses due to random material property, varies at the inner surfaces of the pipe in the direction outward and becomes nearly uniform up to the outer surfaces of the pipe. While the 3rd principal stress shows fluctuation of stresses at the inner surfaces of the pipe and becomes almost uniform elsewhere towards the outside surfaces. Overall, the values of the three principal stresses have increased significantly from using the mean-values to random distribution of Young's modulus. In this particular solution, it clearly shows that the randomness of material property affects significantly the structural responses of the concrete wastewater pipe considered. But the results above cannot be taken as a conclusion for the whole problem since it only represents a particular solution of the problem.

The variation of the three principal stresses at any point of the concrete pipe can also be plotted if desired. As shown in Figure 5.2, node 806 was selected for a representative point of the pipe. Figure 5.15 shows a graph of the probability density versus the variation of the principal stresses at the selected node. The probability distribution function (pdf) above for the principal stresses at node 806 was generated from a simple statistical counting of the sample solutions. The figure shows a Gaussian distribution of the principal stresses. In addition, different shapes and sizes of distribution are shown. The 1st

principal stresses σ_1 in node 806 are spread mostly positive with a maximum probability density of approximately equal to 0.95×10^{-6}. The 2nd principal stresses σ_2 show a narrower distribution and generally positive values with a maximum probability density of about 1.7×10^{-6}. Lastly, the 3rd principal stresses σ_3 have mostly negative values and spreads widely and a maximum probability density of about 0.67×10^{-6}. Note that the figure shows only a solution for a particular node or point in the pipe.

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Chapter 6
Summary and Conclusion

The standard FEM is a very powerful tool used to simulate and analyze virtually all engineering problems. The method assumes that all the given values are deterministic in nature taken as mean values. As a result, FEM solution represents the mean-values of the true values being sought. In reality, most of the engineering analysis involves some form of uncertainties. These uncertainties can be encountered in the material properties, geometry or boundary conditions/loadings. If uncertainties dominate the engineering problem, then a simple finite element analysis is not sufficient for the solution of the said problem. The real challenge to the engineers is on how to solve these problems incorporating the randomness of the system involved. Over the years, to solve stochastic linear systems engineers modified the conventional FEM into SFEM. The SFEM can be thought of as a generalization of FEM by integrating the random variations of the system involved.

In the SFEM, the randomness of the system involved can be integrated into the deterministic PDE's by adding not only a function of the medium domain $D \subset \mathbb{R}^n$, but also a function of probability space $(\Omega, \mathcal{F}, P)$. In this way, the PDE's will become SPDE's where all the mathematical terms are described in stochastic fields by random variables.

In this thesis, only the randomness of the material properties was considered. The thermal conductivity of a material for instance can be defined as $k = k(\boldsymbol{x}, \theta)$ where $\boldsymbol{x}$ is a function of the medium domain and θ is a function of the probability space. The coefficient $k(\boldsymbol{x}, \theta)$ can be defined by its two lowest statistical moments, such as the expectation and

covariance. It is then expanded by the F-K-L representation of stochastic processes. The equivalent SPDE system of any random media is then discretized to achieve a stochastic system of linear algebraic equations using the standard finite element.

Over the years, several methods have been developed to solve this stochastic system of linear algebraic equations, namely: Monte Carlo method; Neumann expansion method; perturbation method; polynomial chaos expansion method; and recently, the joint diagonalization method. In this thesis, a new algorithm called the joint diagonalization strategy is investigated to a particular problem and compared the results to the existing methods. This algorithm is integrated into a SFEM framework, implemented in MATLAB and tested to solve typical engineering problems such as the steady-state heat conduction and the elastostatics in a concrete wastewater pipe. The algorithm is monotonously decreasing in each iterations which ensures the convergence of the method. Further, the joint tridiagonalization algorithm is also under development, this is somewhat similar to the joint diagonalization algorithm but due to time constraint some problems are still to be worked out such as the convergence of the method.

In conclusion, the joint diagonalization strategy is better than the Neumann expansion method and in good agreement to the Monte Carlo method. In terms of relative error, the joint diagonalization algorithm is more accurate than the Neumann expansion method. The Jacobi-like algorithm for multiple real symmetric matrices reduces to the classical Jacobi algorithm for a single real symmetric matrix and the method gives the exact solution in a simple case. Further, the solution for the deterministic equation system (4.2) can be regarded as a special case of the solver for the more general stochastic equation system (4.3). The joint diagonalization can only be approximately achieved except if all the real symmetric matrices $\boldsymbol{A}_i$ $(i = 1, \cdots, n)$ have exactly the same eigenstructure. This means that the approximate result is essentially an average eigenstructure that minimizes all the off-diagonal entries measured by $\sum_{k=1}^{n} off(\boldsymbol{A}_k)$. Lastly, the Jacobi-like joint diagonalization procedure contributes the major computational cost of the new approach which is proportional to the total number of matrices. This also implies that the algorithm can be easily parallelized and the total computational cost is proportional to the total number of random variables in the system.

Lastly, as shown in the numerical examples in Chapter 5, it clearly shows that the randomness of material property affects significantly the structural responses of the concrete wastewater pipe. Hence, the randomness of material properties should be considered in analyzing engineering structures composed of random media. If uncertainties dominate the engineering problem, then a SFEM should be used to simulate and analyze the engineering system. However, computational cost is expected to increase in SFEM compared to FEM. In the future, this will not be an issue as the computer power and data storage are rapidly increasing.

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